CSE 107 10-17-25

Supplemental Lecture

<u>Ex</u> The hat Problem.

Suppose n people throw their hats into a box, then each picks a random hat from the box. This implies each assignment

$$\{\text{hats}\} \longrightarrow \{\text{People}\}$$

is equally likely. Let X = # of People who get their own hat back.

Goal: find $E[X]$

Observe $X = X_1 + X_2 + \cdots + X_n$, where each X_i is Bernoulli with $p = \frac{1}{n}$.

$$X_i = \begin{cases} 1 & \text{if } i^{th} \text{ Person gets hat back} \\ 0 & \text{if not} \end{cases}$$

so

$$E[X_i] = \frac{1}{n}$$

and

$$E[X] = E[X_1 + \cdots + X_n]$$

$$= E[X_1] + \cdots + E[X_n]$$

$$= \frac{1}{n} + \cdots + \frac{1}{n}$$

$$= n \cdot \frac{1}{n} = \boxed{1}$$

2.6 Conditioning

Given a r.v.

$$X : \Omega \to \mathbb{R}$$

any valid probability law $\mathbb{P}(\cdot)$ can be used to define the PMF

$$p_X(x) = \mathbb{P}(X = x)$$

This includes conditional probability laws $\mathbb{P}(\cdot \mid A)$ for $A \subseteq \Omega$ with $\mathbb{P}(A) > 0$.

<u>Defn</u>

Let X be a r.v., $A \subseteq \Omega$ with $P(A) > 0$. The <u>conditional PMF</u> of X given A is

$$\boxed{\begin{aligned} p_{X|A}(x) &= P(\{X=x\} \mid A) \\ &= \frac{P(\{X=x\} \cap A)}{P(A)} \end{aligned}}$$

<u>note</u>: $\{X=x\} \cap A$ are disjoint for different $x \in \text{range}(X)$, so by the total Prob. thm.

$$P(A) = \sum_x P(\{X=x\} \cap A)$$

Thus

$$\sum_x P_{X|A}(x) = \sum_x \frac{P(\{X=x\}\cap A)}{P(A)}$$

$$= \frac{\sum_x P(\{X=x\}\cap A)}{P(A)}$$

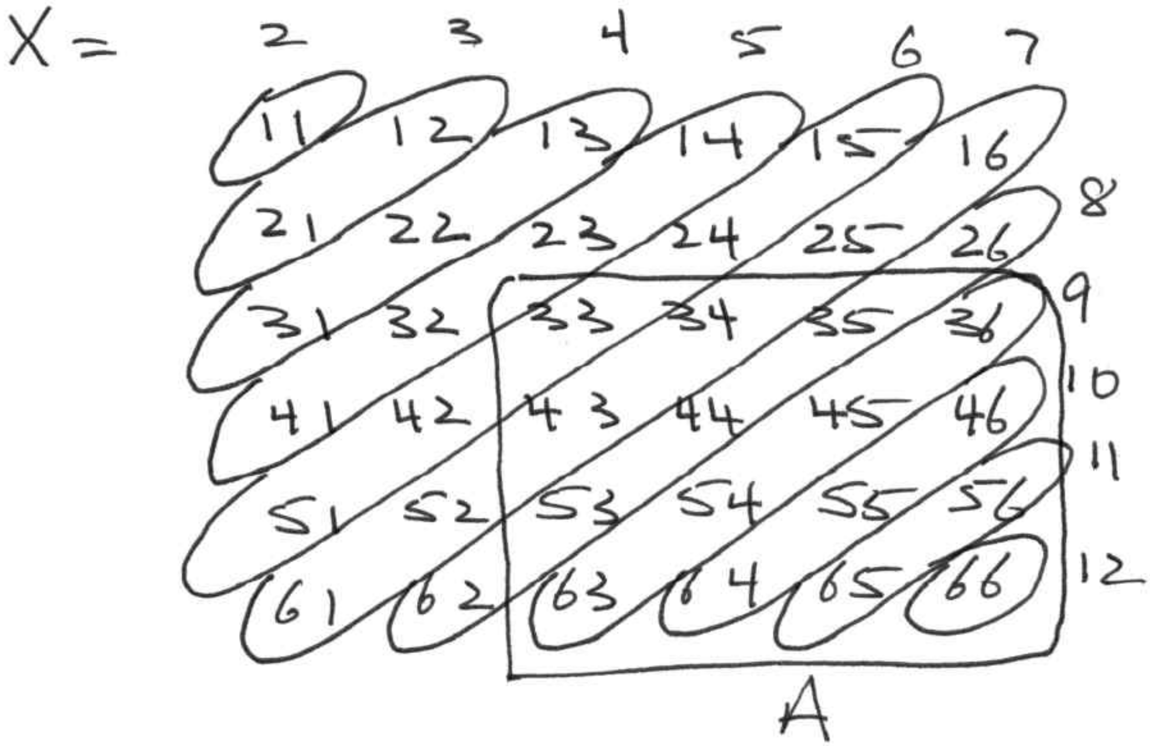
$$= \frac{P(A)}{P(A)} = 1$$

so $P_{X|A}(x)$ is a valid PMF.

Ex

Roll 2 independent, fair, 6-sided dice, and let X = sum of dice

so $\text{range}(X) = \{2,3,4,5,6,7,8,9,10,11,12\}$



PMF of X is:

$$P_X(k) = \begin{cases} 1/36 & \text{if } k = 2, 12 \\ 2/36 & \text{if } k = 3, 11 \\ 3/36 & \text{if } k = 4, 10 \\ 4/36 & \text{if } k = 5, 9 \\ 5/36 & \text{if } k = 6, 8 \\ 6/36 & \text{if } k = 7 \\ 0 & \text{otherwise} \end{cases}$$

let $A = \{\min \geq 3\}$, $P(A) = \frac{16}{36}$.

<u>note</u>: $P_{X|A}(7) = \dfrac{P(\{X=7\} \cap A)}{P(A)} = \dfrac{2/36}{16/36} = \dfrac{2}{16}$

so

$$P_{X|A}(k) = \begin{cases} 1/16 & \text{if } k = 6, 12 \\ 2/16 & \text{if } k = 7, 11 \\ 3/16 & \text{if } k = 8, 10 \\ 4/16 & \text{if } k = 9 \\ 0 & \text{otherwise} \end{cases}$$

<u>Ex</u> flip a coin with $P(H) = p$ until the 1st head (independent flips).
Let $X = \#$ flips. so is Geometric(p).

$$P_X(k) = \begin{cases} (1-p)^{k-1} \cdot p & \text{if } k = 1, 2, 3, \ldots\ldots \\ 0 & \text{otherwise} \end{cases}$$

let $A = \{ 1^{st} \text{ head is within } n \text{ flips} \}$

$= \{ X \leq n \}$

$$P(A) = \sum_{k=1}^{n} p_X(k)$$

$$= \sum_{k=1}^{n} (1-p)^{k-1} \cdot p$$

$$= p \cdot \sum_{k=0}^{n-1} (1-p)^k \quad \{ k \leftarrow k+1$$

$$= \cancel{p} \cdot \frac{1-(1-p)^n}{\cancel{1-(1-p)}} = 1-(1-p)^n$$

another way

$$P(A) = 1 - P(A^c)$$

$$= 1 - P(1^{st}\ n \text{ flips are tails})$$

$$= 1-(1-p)^n$$

now

$$P_{X|A}(k) = \frac{P(\{X=k\} \cap A)}{P(A)}$$

non-empty iff $1 \leq k \leq n$ and $= \{X=k\}$

$$= \frac{P(X=k)}{P(A)}$$

$$= \frac{(1-p)^{k-1} \cdot p}{1-(1-p)^n}$$

i.e.

$$P_{X|A}(k) = \begin{cases} \dfrac{(1-p)^{k-1} \cdot p}{1-(1-p)^n} & \text{if } k = 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

Now let Y be another r.v.

note that $p_Y(y) = P(Y=y) > 0$ for

any $y \in \text{range}(Y)$. we specialize

to A of form: $A = \{Y=y\}$

Defn

The <u>conditional PMF of X given Y</u> is

$$\boxed{p_{X|Y}(x|y) = P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}}$$

observe that

$$\boxed{P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}}$$

Thus

$$\boxed{P_{X,Y}(x,y) = P_Y(y) \cdot P_{X|Y}(x|y)}$$

and also

$$\boxed{P_{X,Y}(x,y) = P_X(x) \cdot P_{Y|X}(y|x)}$$

Ex

A professor answers questions wrongly with probability 1/4, independent of other questions. The number of questions is 0, 1 or 2 with equal probability. Let

$$X = \#\text{ questions asked}$$

$$Y = \#\text{ wrong answers}$$

find the joint PMF $P_{X,Y}(x,y)$

We have

$$P_X(x) = \begin{cases} \frac{1}{3} & \text{if } x = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

also

$$P(Y=0 \mid X=0) = 1$$

$$P(Y=0 \mid X=1) = \frac{3}{4}$$

$$P(Y=1 \mid X=1) = \frac{1}{4}$$

$$P(Y=0 \mid X=2) = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$$

$$P(Y=1 \mid X=2) = \frac{3}{4}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{3}{4} = \frac{6}{16} = \frac{3}{8}$$

$$P(Y=2 \mid X=2) = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$$

Thus

$$P_{Y|X}(y|x) = \begin{cases} 1 & x=0,\ y=0 \\ 3/4 & x=1,\ y=0 \\ 1/4 & x=1,\ y=1 \\ 9/16 & x=2,\ y=0 \\ 3/8 & x=2,\ y=1 \\ 1/16 & x=2,\ y=2 \\ 0 & \text{otherwise} \end{cases}$$

Therefore

$$P_{X,Y}(x,y) = \begin{cases} 1/3 & x=0,\ y=0 \\ 1/4 & x=1,\ y=0 \\ 1/12 & x=1,\ y=1 \\ 3/16 & x=2,\ y=0 \\ 1/8 & x=2,\ y=1 \\ 1/48 & x=2,\ y=2 \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$P_X(x) \qquad \underbrace{\begin{array}{|c|c|c|} \hline 1/3 & 1/3 & 1/3 \\ \hline 0 & 1 & 2 \end{array}}_{x}$$

$$P_{Y|X}(y|x) \qquad y \begin{cases} 2 \\ 1 \\ 0 \end{cases} \begin{array}{|c|c|c|} \hline & & 1/16 \\ \hline & 1/4 & 3/8 \\ \hline 1 & 3/4 & 9/16 \\ \hline \end{array}$$
$$\underbrace{\quad 0 \quad 1 \quad 2 \quad}_{x}$$

$$P_{X,Y}(x,y) \qquad y \begin{cases} 2 \\ 1 \\ 0 \end{cases} \begin{array}{|c|c|c|} \hline & & 1/48 \\ \hline & 1/12 & 1/8 \\ \hline 1/3 & 1/4 & 3/16 \\ \hline \end{array} \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} \ y \begin{cases} 2 \\ 1 \\ 0 \end{cases} \begin{array}{|c|} \hline 1/48 \\ \hline 10/48 \\ \hline 37/48 \\ \hline \end{array}$$
$$\underbrace{\quad 0 \quad 1 \quad 2 \quad}_{x}$$

$P_Y(y)$

We now have

$$P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

Thus

$P_{X|Y}(x|y)$

y \ x	0	1	2
2			1
1		2/5	3/5
0	16/37	12/37	9/37

- see summary of facts on P. 103 in text: conditional PMFs.

Conditional Expectation

Let X be a r.v., $A \subseteq \Omega$ with $P(A) > 0$. Define

$$\boxed{E[X|A] = \sum_x x\, p_{X|A}(x)}$$

expected value rule

$$\boxed{E[g(X)|A] = \sum_x g(x)\, p_{X|A}(x)}$$

we can condition on $A = \{Y = y\}$ provided $P(Y=y) > 0$.

$$\boxed{E[X|Y=y] = \sum_x x\, p_{X|Y}(x|y)}$$

Total Expectation Theorem

Let $A_1, A_2, \ldots, A_n$ be a partition of Ω.

Then

$$\boxed{E[X] = \sum_{i=1}^{n} P(A_i) \cdot E[X \mid A_i]}$$

Proof

by the tot. Prob. thm, we have

$$p_X(x) = \sum_{i=1}^{n} P(A_i) \cdot p_{X|A_i}(x)$$

Multiply both sides by x, and sum over $x \in \text{range}(X)$, to get

$$E[X] = \sum_x x \, p_X(x)$$

$$= \sum_x x \sum_{i=1}^{n} P(A_i) \cdot p_{X|A_i}(x)$$

$$= \sum_x \sum_{i=1}^{n} P(A_i) \cdot x \, p_{X|A_i}(x)$$

$$= \sum_{i=1}^{n} P(A_i) \cdot \sum_x x \, p_{X|A_i}(x)$$

$$= \sum_{i=1}^{n} P(A_i) \, E[X|A_i]$$

$\blacksquare$

- See similar identities on P. 104

Mean & Variance of Geometric r.v.

Recall, if X is Geometric(p), then

$$P_X(k) = \begin{cases} (1-p)^{k-1} \cdot p & \text{if } k = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Then

$$E[X] = \sum_{k=1}^{\infty} k (1-p)^{k-1} \cdot p$$

$$E[X^2] = \sum_{k=1}^{\infty} k^2 (1-p)^{k-1} \cdot p$$

and

$$Var(X) = E[X^2] - E[X]^2$$

We will use total expectation to find $E[X]$, $E[X^2]$.

Let $A_1 = \{X = 1\} = \{1^{st} \text{ flip is head}\}$

$A_2 = \{X > 1\} = \{1^{st} \text{ flip is tail}\}$.

Then $P(A_1) = p$, and $P(A_2) = 1-p$.

observe $E[X|A_1] = 1$, and

$$E[X|A_2] = E[1+X] = 1 + E[X]$$

Thus

$$E[X] = P(A_1) \cdot E[X|A_1] + P(A_2) \cdot E[X|A_2]$$

$$E[X] = p \cdot 1 + (1-p)(1+E[X])$$

$$\cancel{E[X]} = \cancel{p} + 1 - \cancel{p} + \cancel{E[X]} - pE[X]$$

$$0 = 1 - pE[X]$$

$$\therefore \quad E[X] = \boxed{\frac{1}{p}}$$

Similarly

$$E[X^2 | A_1] = 1$$

and

$$E[X^2 | A_2] = E[(1+X)^2]$$

$$= E[1 + 2X + X^2]$$

$$= 1 + 2\underbrace{E[X]}_{\frac{1}{p}} + E[X^2]$$

Therefore

$$E[X^2] = P(A_1) \cdot E[X^2 | A_1] + P(A_2) E[X^2 | A_2]$$

$$'' \quad = p \cdot 1 + (1-p)\left(1 + 2 \cdot \frac{1}{p} + E[X^2]\right)$$

$$\cancel{E[X^2]} = \cancel{p} + (1-\cancel{p}) + \frac{2-2p}{p} + (\cancel{1}-p) E[X^2]$$

$$0 = 1 + \frac{2-2p}{p} - p\, E[X^2]$$

$$\therefore\ p\,E[X^2] = 1 + \frac{2-2p}{p} = \frac{p+2-2p}{p} = \frac{2-p}{p}$$

$$\therefore\ E[X^2] = \frac{2-p}{p^2} = \boxed{\frac{2}{p^2} - \frac{1}{p}}$$

Therefore

$$Var(X) = \left(\frac{2}{p^2} - \frac{1}{p}\right) - \frac{1}{p^2}$$

$$= \frac{1}{p^2} - \frac{1}{p}$$

$$Var(X) = \boxed{\frac{1-p}{p^2}}$$