CSE 107 10-16-25

- Midterm 1 : Thur. 10/23
  - exam : 1:30 - 2:35
  - lecture to follow

Quiz Problem ... Continue

$$(Q_1, Q_2) \Rightarrow E[\text{Payoff}] = V_1 P_1 + V_2 P_1 P_2$$

$$(Q_2, Q_1) \Rightarrow E[\text{Payoff}] = V_2 P_2 + V_1 P_1 P_2$$

We Prefer (Q_1, Q_2) iff

$$V_1 P_1 + V_2 P_1 P_2 > V_2 P_2 + V_1 P_1 P_2$$

$$\therefore \quad v_1 p_1 - v_1 p_1 p_2 > v_2 p_2 - v_2 p_1 p_2$$

$$\therefore \quad v_1 p_1 (1 - p_2) > v_2 p_2 (1 - p_1)$$

$$\therefore \quad \frac{v_1 p_1}{1 - p_1} > \frac{v_2 p_2}{1 - p_2}$$

<u>Exercise</u>

analyze this with 3 questions.

(Thus 6 possible orders)

2.5 Joint PMFs, Multiple r.v.s

Defn

Given two r.v.s X, Y their **Joint PMF** is

$$\boxed{p_{X,Y}(x,y) = P(X=x,\ Y=y)}$$

$$= P(\{X=x\} \cap \{Y=y\})$$

if $A \subseteq \underline{\text{range}}(X) \times \underline{\text{range}}(Y)$,

then

$$P\big((x,y) \in A\big) = \sum_{(x,y)\in A} p_{X,Y}(x,y)$$

Ex. Suppose

$$\text{range}(X) = \{1, 2, 3, 4\}$$

$$\text{range}(Y) = \{1, 2, 3\}$$

let Joint PMF be

Y \ X	1	2	3	4	$p_Y(y)$
3	1/20	1/20	2/20	1/20	5/20
2	2/20	3/20	3/20	1/20	9/20
1	1/20	1/20	1/20	3/20	6/20
$p_X(x)$	4/20	5/20	6/20	5/20	

$p_{X,Y}(x,y)$

let $y=3$. since

$$\{X=1\}\cup\{X=2\}\cup\{X=3\}\cup\{X=4\}=\Omega$$

is a Partition of Ω. By total Prob. thm.

$$P(Y=3)=P(X=1,Y=3)+P(X=2,Y=3)$$
$$+P(X=3,Y=3)+P(X=4,Y=3)$$

$$P_Y(3)=\sum_{x=1}^{4}P_{X,Y}(x,3)=\frac{5}{20}$$

Thus

$$\boxed{P_X(x) = \sum_y P_{X,Y}(x,y)} \quad \text{marginal}$$

and

$$\boxed{P_Y(y) = \sum_x P_{X,Y}(x,y)} \quad \text{marginal}$$

Functions of multiple r.v.s

Let $Z = g(X,Y)$ where

$$g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

Then

$$p_Z(z) = \sum_{\{(x,y) \mid g(x,y) = z\}} p_{X,Y}(x,y)$$

<u>Generalized Expected Value Rule</u>

$$\boxed{E[g(X,Y)] = \sum_x \sum_y g(x,y) \cdot p_{X,Y}(x,y)}$$

<u>Exercise</u>: Prove this. (See text.)

Ex.

$$E[aX+bY+c] = aE[X] + bE[Y] + c$$

Proof

$$E[aX+bY+c] = \sum_x \sum_y (ax+by+c)\, p_{X,Y}(x,y)$$

$$= a\sum_x \sum_y x\, p_{X,Y}(x,y) + b\sum_x \sum_y y\, p_{X,Y}(x,y)$$

$$+ c\sum_x \sum_y p_{X,Y}(x,y)$$

$$= a\sum_x x\left(\sum_y p_{X,Y}(x,y)\right) + b\sum_y y\left(\sum_x p_{X,Y}(x,y)\right) + c$$

$$= a\sum_x x\, p_X(x) + b\sum_y y\, p_Y(y) + c$$

$$= a\,E[X] + b\,E[Y] + c$$

$\blacksquare$

<u>Ex</u> (previous example)

$$E[X+3Y] = E[X] + 3E[Y]$$

$$= \left(1\cdot\frac{4}{20} + 2\cdot\frac{5}{20} + 3\cdot\frac{6}{20} + 4\cdot\frac{5}{20}\right)$$

$$+ 3\left(1\cdot\frac{6}{20} + 2\cdot\frac{9}{20} + 3\cdot\frac{5}{20}\right)$$

$$= \cdots = \frac{169}{20}$$

more than 2 r.v.s:

$$p_{X,Y,Z}(x,y,z) = P(X=x, Y=y, Z=z)$$

Joint PMF of r.v.s X, Y, Z

have 3, 2-variable marginals:

$$p_{X,Y}(x,y) = \sum_z p_{X,Y,Z}(x,y,z)$$

$$p_{X,Z}(x,z) = \sum_y p_{X,Y,Z}(x,y,z)$$

$$p_{Y,Z}(y,z) = \sum_x p_{X,Y,Z}(x,y,z)$$

also have 3 1-variable marginals:

$$P_X(x) = \sum_y \sum_z P_{X,Y,Z}(x,y,z)$$

$$P_Y(y) = \ldots$$

$$P_Z(z) = \ldots$$

Expected value rule:

$$\boxed{E[g(X,Y,Z)] = \sum_{x,y,z} g(x,y,z) P_{X,Y,Z}(x,y,z)}$$

Suppose we have n r.v.s

$$X_1, X_2, \ldots, X_n$$

then

$$E[a_1X_1 + a_2X_2 + \cdots + a_nX_n + b]$$

$$= a_1 E[X_1] + a_2 E[X_2] + \cdots + a_n E[X_n] + b$$

<u>Exercise</u> Prove this by induction on n.

The Mean of the Binomial

Let X be Binomial(n, p).

Recall

$$P_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & (0 \le k \le n) \\ 0 & \text{otherwise} \end{cases}$$

The mean is

$$E[X] = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

<u>note</u>

$$X = X_1 + X_2 + \cdots + X_n$$

where each X_i is Bernoulli(p).

Recall $E[X_i] = 1 \cdot p + 0 \cdot (1-p) = p$

Thus

$$E[X] = E[X_1 + \cdots + X_n]$$

$$= E[X_1] + \cdots + E[X_n]$$

$$= p + \cdots + p$$

$$= np$$

we will show (later) that

$$\mathrm{Var}(X) = np(1-p)$$

next ... heat problem ...