CSE 107 10-14-25

Variance in terms of moments:

$$Var(X) = E[X^2] - (E[X])^2$$

<u>notation</u> $\mu = E[X]$

<u>Proof</u>

$$Var(X) = E[(X-\mu)^2]$$

$$= \sum_x (x-\mu)^2 p_X(x)$$

$$= \sum_x (x^2 - 2\mu x + \mu^2) p_X(x)$$

$$= \sum_{x} \left(x^2 p_X(x) - 2\mu x p_X(x) + \mu^2 p_X(x) \right)$$

$$= \sum_{x} x^2 p_X(x) - 2\mu \sum_{x} x p_X(x) + \mu^2 \sum_{x} p_X(x)$$

$$= E[X^2] - 2\mu \underbrace{E[X]}_{\mu} + \mu^2$$

$$= E[X^2] - \mu^2$$

$$= E[X^2] - \left(E[X]\right)^2$$

$\blacksquare$

Ex. Recall if $Y = aX + b$, then

$$E[Y] = E[aX+b] = aE[X] + b.$$

Find $Var(Y)$. first find 2^{nd} moment of Y

$$E[Y^2] = E[(aX+b)^2]$$

$$= \sum_x (ax+b)^2 p_X(x)$$

$$= \sum_x (a^2x^2 + 2abx + b^2) p_X(x)$$

$$= a^2 \sum_x x^2 p_X(x) + 2ab \sum_x x\, p_X(x) + b^2 \sum_x p_X(x)$$

$$a^2 \cdot E[X^2] + 2ab E[X] + b^2$$

now Variance!

$$Var(Y) = Var(aX+b)$$

$$= E[(aX+b)^2] - E[aX+b]^2$$

$$= a^2 E[X^2] + \cancel{2ab E[X] + b^2}$$

$$- (a^2 E[X]^2 + \cancel{2ab E[X] + b^2})$$

$$= a^2 (E[X^2] - E[X]^2)$$

$$= a^2 Var(X)$$

Summary

- $E[aX+b] = aE[X]+b$
- $Var(aX+b) = a^2 Var(X)$

<u>note</u>:

- if $X = \text{const}$, then $Var(X) = 0$
- <u>exercise</u> prove that

$$Var(X) = 0 \Rightarrow X = \text{const}$$

- Bernoulli R.V.

$$P_X(k) = \begin{cases} p & k=1 \\ 1-p & k=0 \end{cases}$$

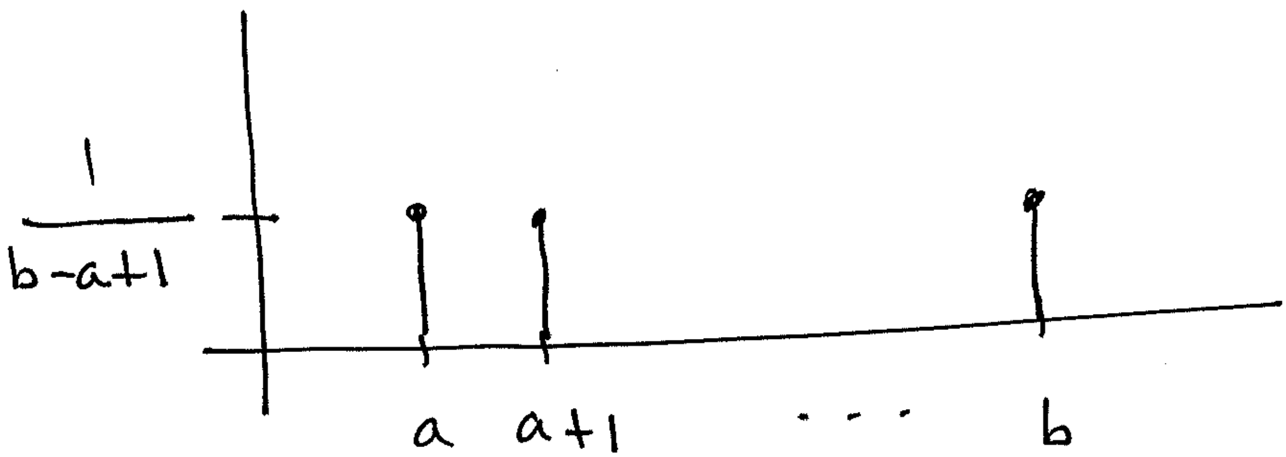
$$E[X] = 1 \cdot p + 0 \cdot (1-p) = \boxed{p}$$

$$E[X^2] = 1^2 \cdot p + 0^2 \cdot (1-p) = p$$

$$Var(X) = E[X^2] - E[X]^2$$

$$= p - p^2 = \boxed{p(1-p)}$$

- Discrete Uniform on $\{a, a+1, \dots, b\}$



$$p_X(k) = \begin{cases} \frac{1}{b-a+1} & \text{if } k = a, a+1, \dots b \\ 0 & \text{otherwise} \end{cases}$$

so

$$E[X] = \sum_{k=a}^{b} k \cdot \frac{1}{b-a+1}$$

$$= \frac{1}{b-a+1} \cdot \left(\sum_{k=1}^{b} k - \sum_{k=1}^{a-1} k \right)$$

$$= \frac{1}{b-a+1} \cdot \left(\frac{b(b+1)}{2} - \frac{a(a-1)}{2} \right)$$

$$= \frac{1}{2} \cdot \left(\frac{1}{b-a+1} \right) \left(b^2 + b - a^2 + a \right)$$

$$= \frac{1}{2} \left(\frac{1}{b-a+1} \right) \left(\underbrace{(b^2 - a^2)}_{(b-a)(b+a)} + (b+a) \right)$$

$$= \frac{1}{2} \left(\frac{1}{\cancel{b-a+1}} \right) (b+a) \cancel{(b-a+1)}$$

$$= \boxed{\frac{a+b}{2}}$$

To compute $Var(X)$, set

$$Y = X + (1-a)$$

so $Var(Y) = Var(X+(1-a)) = Var(X)$

let $n = b-a+1$. Then

$$X = a \Rightarrow Y = a + (1-a) = 1$$

$$X = b \Rightarrow Y = b + (1-a) = n$$

Thus $E[Y] = \frac{n+1}{2}$, and

$$E[Y^2] = \sum_{k=1}^{n} k^2 \cdot \frac{1}{n} = \frac{1}{n} \cdot \sum_{k=1}^{n} k^2$$

$$= \frac{1}{\cancel{n}} \cdot \frac{\cancel{n}(n+1)(2n+1)}{6}$$

Thus

$$\mathrm{Var}(X) = \mathrm{Var}(Y)$$

$$= \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$$

$$= (n+1)\left(\frac{2n+1}{6} - \frac{n+1}{4}\right) = \underbrace{\cdots}_{\text{exercise}} =$$

$$= \frac{(n+1)(n-1)}{12}$$

$$= \frac{(b-a)(b-a+2)}{12}$$

• Poisson r.v. with Param. λ

$$P_X(k) = \begin{cases} e^{-\lambda} \cdot \dfrac{\lambda^k}{k!} & (k = 0, 1, 2, \ldots) \\ 0 & \text{otherwise} \end{cases}$$

will show:

(1) $E[X] = \lambda$ ✓

(2) $E[X^2] = \lambda^2 + \lambda$ ✓

Thus $Var(X) = (\lambda^2 + \lambda) - \lambda^2 = \lambda$

Proof of (1)

$$E[X] = \sum_{k=\cancel{0}1}^{\infty} k \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \cdot \sum_{k=1}^{\infty} k \cdot \frac{\lambda^k}{k!} \qquad (*)$$

$$= \lambda e^{-\lambda} \cdot \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

$$= \lambda e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \qquad \left\{ \begin{array}{l} \text{replace} \\ k \leftarrow k+1 \end{array} \right.$$

$$= \lambda \cancel{e^{-\lambda}} \cdot \cancel{e^{\lambda}} = \lambda$$

Proof of (2)

$$E[X^2] = \sum_{k=\cancel{0}1}^{\infty} k^2 \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= \lambda e^{-\lambda} \cdot \sum_{k=1}^{\infty} k \frac{\lambda^{k-1}}{(k-1)!}$$

$$= \lambda e^{-\lambda} \cdot \sum_{k=0}^{\infty} (k+1) \frac{\lambda^k}{k!} \quad \left\{ \begin{array}{l} \text{replace} \\ k \leftarrow k+1 \end{array} \right.$$

$$= \lambda e^{-\lambda} \left(\sum_{k=\cancel{0}1}^{\infty} k \cdot \frac{\lambda^k}{k!} + \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right)$$

$$= \lambda \left(e^{-\lambda} \cdot \sum_{k=1}^{\infty} k \cdot \frac{\lambda^k}{k!} + e^{-\lambda} \cdot e^{\lambda} \right)$$

$$= \lambda (\lambda + 1)$$

$$= \lambda^2 + \lambda$$

Ex. The Quiz Problem

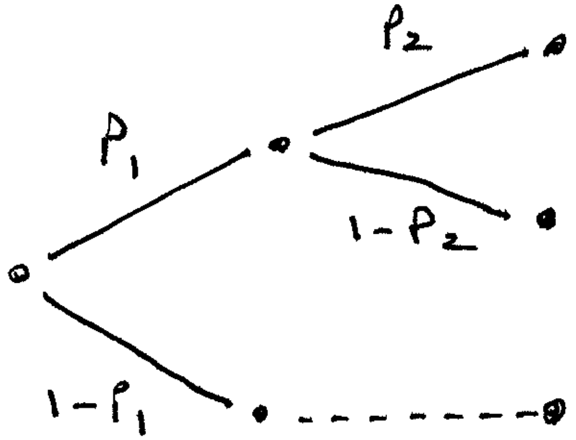
A game consists of 2 questions

	Q_1	Q_2
P(correct)	p_1	p_2
Payoff (\$)	v_1	v_2

1^{st} wrong answer terminates the game. You can answer in either order

$$(Q_1, Q_2) \text{ or } (Q_2, Q_1)$$

Find PMF of total payoff X for order (Q_1, Q_2), then find $E[X]$



Prob	Payoff
$p_1 p_2$	$v_1 + v_2$
$p_1(1-p_2)$	v_1
$(1-p_1)$	0

$$P_X(x) = \begin{cases} p_1 p_2 & \text{if } x = v_1 + v_2 \\ p_1(1-p_2) & \text{if } x = v_1 \\ (1-p_1) & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

so

$$E[X] = (v_1 + v_2) p_1 p_2 + v_1 p_1 (1 - p_2)$$

$$= \cancel{v_1 p_1 p_2} + v_2 p_1 p_2 + v_1 p_1 - \cancel{v_1 p_1 p_2}$$

$$= v_1 p_1 + v_2 p_1 p_2$$

let Y be payoff for (Q_2, Q_1)

$$E[Y] = v_2 p_2 + v_1 p_1 p_2$$

$\vdots$

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