CSE 107 10-10-25

Supplemental Lecture

<u>Ex</u>

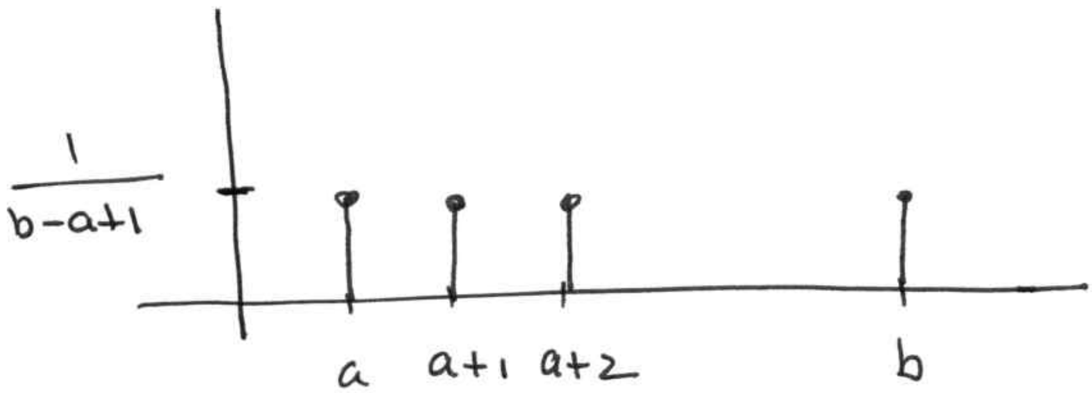
let $n=200$, $\lambda=2$, $p=\frac{\lambda}{n}=.01$. The Prob. of $k=4$ successes in 200 indep. Bernoulli trials, with .01 Prob. of success is

$$\binom{200}{4}(.01)^4(.99)^{196} = \underbrace{.0902}197..$$

This is well approximated by

$$e^{-2}\cdot\frac{2^4}{4!} = \underbrace{.0902}235..$$

The Discrete Uniform Random Variable



ints in range $\{a, a+1, \ldots, b\}$ is $(b-a+1)$. The PMF is

$$P_X(k) = \begin{cases} \frac{1}{b-a+1} & \text{if } k = a, a+1, \ldots, b \\ 0 & \text{otherwise} \end{cases}$$

2.3 Functions of a random variable

Given a r.v. X, we can transform it by composition with a function

$$g : \mathbb{R} \to \mathbb{R}.$$

Recall $X : \Omega \to \mathbb{R}$, so we have

$$Y = g(X) = g \circ X : \Omega \to \mathbb{R}$$

is itself another random variable.

<u>Ex.</u> let X have PMF

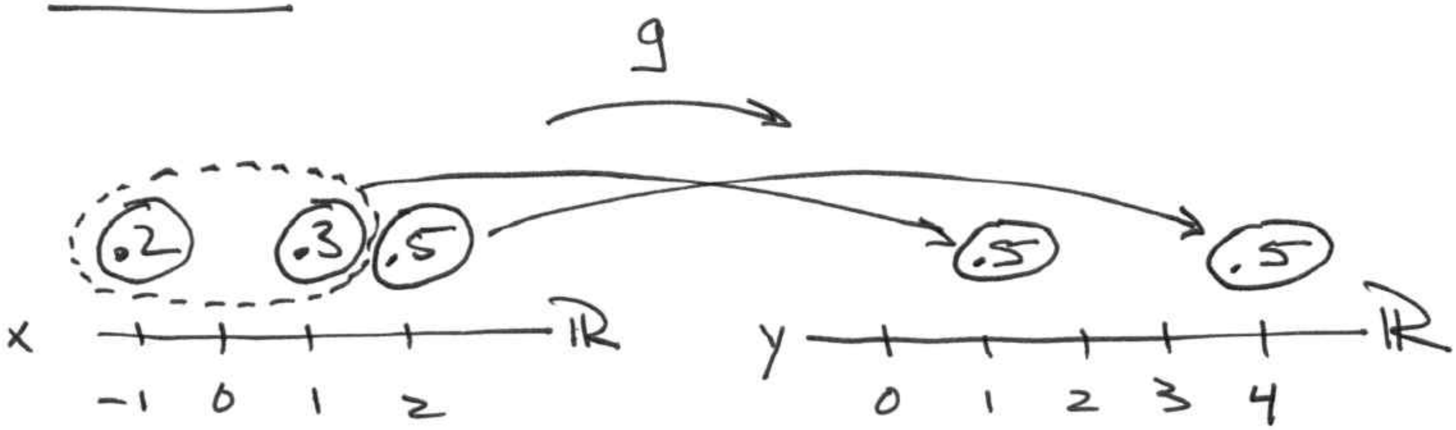
$$P_X(x) = \begin{cases} .2 & \text{if } x = -1 \\ .3 & \text{if } x = 1 \\ .5 & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

Let $Y = X^2$, i.e. $g(x) = x^2$.

Goal: find PMF of Y

$$
\begin{aligned}
P_Y(y) &= P(Y=y) \\
&= P(X^2=y) \\
&= P(X=\pm\sqrt{y}) \\
&= P(X=\sqrt{y} \text{ or } X=-\sqrt{y}) \\
&= P(X=\sqrt{y}) + P(X=-\sqrt{y}) \\
&= \begin{cases} \cancel{.2 \text{ if } \sqrt{y}=-1} \\ .3 \quad \sqrt{y}=1 \\ .5 \quad \sqrt{y}=2 \end{cases} + \begin{cases} .2 \text{ if } -\sqrt{y}=-1 \\ \cancel{.3 \quad -\sqrt{y}=1} \\ \cancel{.5 \quad -\sqrt{y}=2} \end{cases} \\
&= \begin{cases} .3 & \text{if } y=1 \\ .5 & \text{if } y=4 \\ 0 & \text{otherwise} \end{cases} + \begin{cases} .2 & \text{if } y=1 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} .5 & \text{if } y=1 \\ .5 & \text{if } y=4 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Picture:



2.4 Expectation, Mean, Variance

Defn

The expected value (or mean) of a discrete r.v. X is

$$\boxed{E[X] = \sum_{x \in \mathrm{range}(X)} x \cdot p_X(x)}$$

Ex. Previous ex.

$$E[X] = (-1)(.2) + (1)(.3) + (2)(.5) = \boxed{1.1}$$

$$E[X^2] = E[Y] = 1 \cdot (.5) + 4(.5) = \boxed{2.5}$$

note: $E[X^2] \neq E[X]^2$

Expected Value Rule

$$\boxed{E[g(X)] = \sum_x g(x) p_X(x)}$$

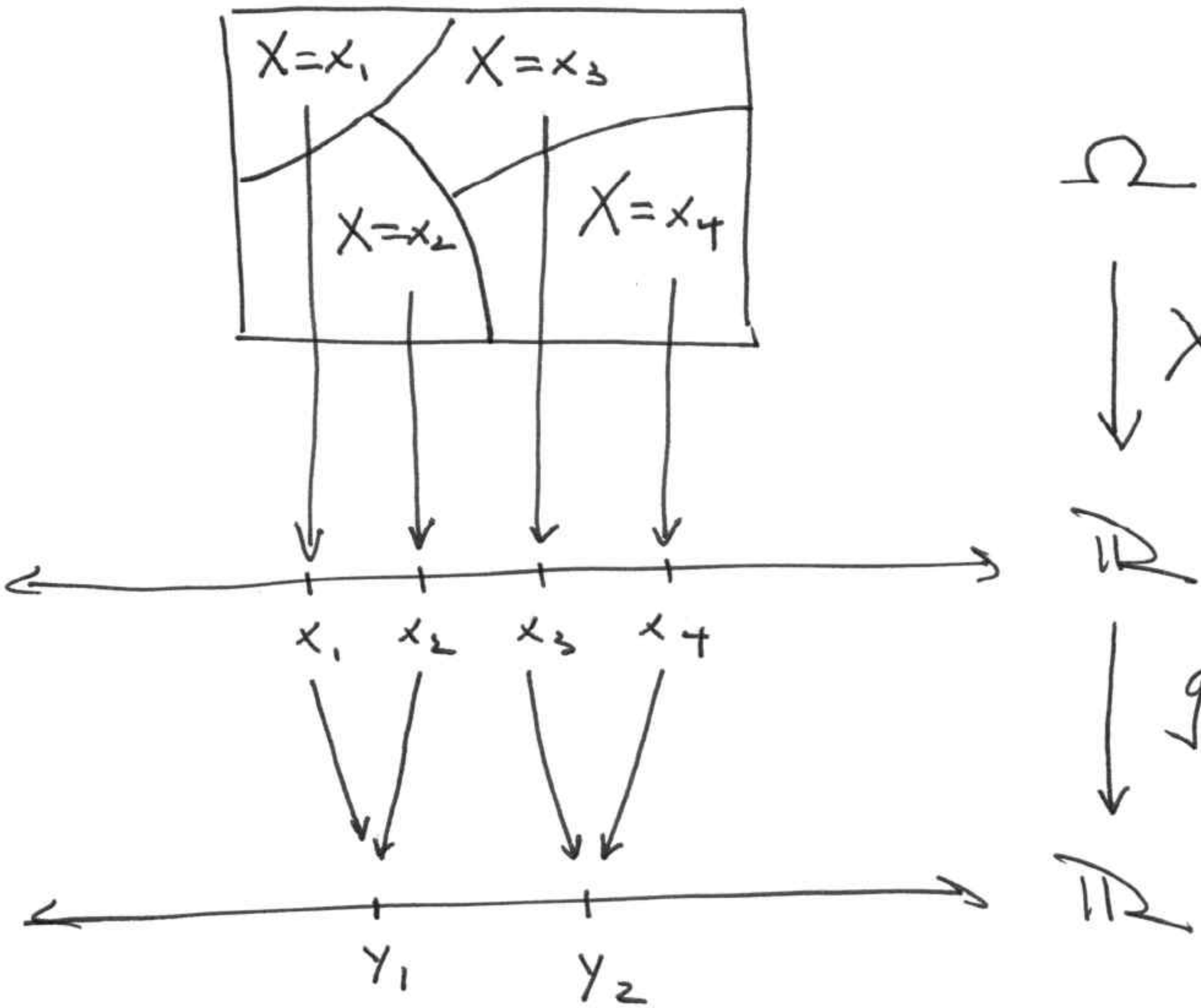
Ex Previous

$$E[X^2] = (-1)^2(.2) + (1)^2(.3) + (2)^2(.5)$$

$$= (.2) + (.3) + 4(.5)$$

$$= 5(.5) = \boxed{2.5}$$

Picture



$$E[Y] = y_1 \cdot P_Y(y_1) + y_2 P_Y(y_2)$$

$$= y_1\left(P_X(x_1)+P_X(x_2)\right) + y_2\left(P_X(x_3)+P_X(x_4)\right)$$

$$=\left(y_1 P_X(x_1) + y_1 P_X(x_2)\right)+\left(y_2 P_X(x_3) + y_2 P_X(x_4)\right)$$

$$= g(x_1)P_X(x_1)+g(x_2)P_X(x_2) + g(x_3)P_X(x_3)+g(x_4)P_X(x_4)$$

$$= \sum_x g(x)\cdot P_X(x)$$

Proof of expected value rule

observe

$$\{Y=y\} = \{\omega \mid Y(\omega)=y\}$$

$$= \{\omega \mid g(X(\omega)) = y\}$$

$$= \bigcup_{\{x \mid g(x)=y\}} \{\omega \mid X(\omega) = x\}$$

$$= \bigcup_{\{x \mid g(x)=y\}} \{X = x\} \quad \begin{bmatrix}\text{disjoint}\\ \text{union}\end{bmatrix}$$

$$\therefore P_Y(y) = P(Y=y) = \sum_{\{x \mid g(x)=y\}} P(X=x)$$

$$= \sum_{\{x \mid g(x)=y\}} P_X(x)$$

now

$$E[Y] = \sum_{y} y \, p_Y(y)$$

$$= \sum_{y} y \sum_{\{x \mid g(x)=y\}} p_X(x)$$

$$= \sum_{y} \sum_{\{x \mid g(x)=y\}} y \, p_X(x)$$

$$= \sum_{y} \sum_{\{x \mid g(x)=y\}} g(x) \, p_X(x)$$

$$= \sum_{x} g(x) \, p_X(x)$$

$\blacksquare$

Ex. $Y = aX + b \quad (a \neq 0)$.

$$E[Y] = E[aX+b]$$

$$= \sum_x (ax+b) p_X(x)$$

$$= \sum_x (ax\, p_X(x) + b\, p_X(x))$$

$$= a\sum_x x\, p_X(x) + b \sum_x p_X(x)$$

$$= aE[X] + b$$

special cases

$b=0$: $\quad E[aX] = aE[X]$

$a=0$: $\quad E[b] = b$

Defn

The n^{th} moment of X $(n=0,1,2,\ldots)$

is

$$E[X^n]=\sum_x x^n\cdot p_X(x)$$

In particular, the 1^{st} moment is $E[X]$, the mean of X.

Defn

The variance of X is the mean of the r.v. $(X-E[X])^2$, i.e.

$$Var(X)=E\left[(X-E[X])^2\right]$$

note $Var(X)\geq 0$ since $(X-E[X])^2\geq 0$

Defn

The <u>standard deviation</u> of X is

$$\sigma_X = \sqrt{Var(X)}$$

<u>Rmk</u> both $Var(X)$ and σ_X are measures of the dispersion of mass around $E[X]$

<u>Ex</u> (Previous)

suppose X has PMF

$$P_X(x) = \begin{cases} .2 & \text{if } x = -1 \\ .3 & \text{"} \quad x = 1 \\ .4 & \text{"} \quad x = 2 \\ 0 & \text{otherwise} \end{cases}$$

we computed

$$E[X] = 1.1 \qquad (1^{st} \text{ moment})$$

$$E[X^2] = 2.5 \qquad (2^{nd} \text{ moment})$$

now

$$Var(X) = E[(X - E[X])^2]$$

$$= \sum_x (x - (1.1))^2 P_X(x)$$

$$= (-1-(1.1))^2(.2) + (1-(1.1))^2 \cdot (.3) + (2-(1.1))^2(.5)$$

$$= \boxed{1.29}$$

and

$$\sigma_X = \boxed{1.1358}$$

Variance in terms of Moments

$$Var(X) = E[X^2] - (E[X])^2$$

Ex. (Previous)

$$Var(X) = (2.5) - (1.1)^2 = \boxed{1.29}$$

Proof next time -----