cse 107 9-26-24

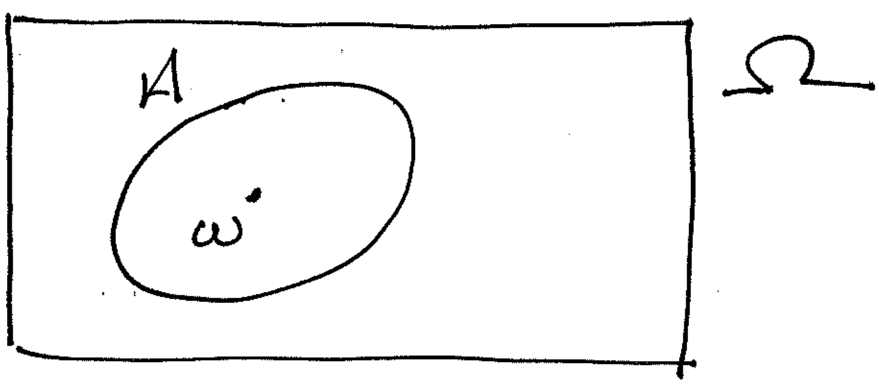
1.1 Sets: Review, read it.

1.2 Probability Models

Defn A Probability model consists of

(1) a set Ω called the sample space. The element $\omega \in \Omega$ are called outcomes, of some experiment. A subset $A \subseteq \Omega$ is called an event.

(2) A <u>Probability Law</u> P assigns to an event $A \subseteq \Omega$ a number $P(A)$ called the <u>Probability of A</u>.



$P(A)$ is the 'likelyhood' that the outcome $\omega \in A$.

Axioms for $P(\cdot)$

(i) $P(A) \geq 0$ for all $A \subseteq \Omega$

(ii) $P(\Omega) = 1$

(iii) a. If $A, B \subseteq \Omega$ are disjoint (i.e. $A \cap B = \phi$), then

- $P(A \cup B) = P(A) + P(B)$

also

- $P(A \cup B \cup C) = P(A) + P(B) + P(C)$ if A, B, C are pairwise disjoint

$\vdots$

we call this <u>finite additivity</u>.

(iii) b. More generally if

$A_1, A_2, A_3, \ldots$ are a sequence of pairwise disjoint events (i.e. if $i \neq j$, then $A_i \cap A_j = \phi$), then

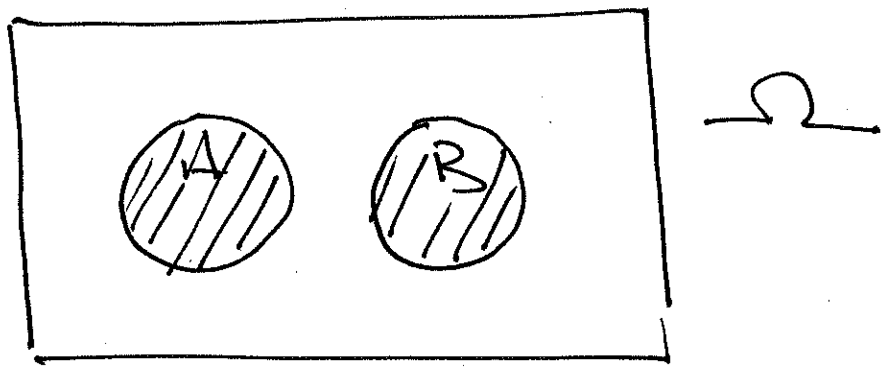
$$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$$

i.e.

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

we call this <u>countable additivity</u>.

2-set version



$$P(A \cup B) = P(A) + P(B).$$

Ex. flip a fair coin

$$\Omega = \{H, T\}$$

Events: $\emptyset, \{H\}, \{T\}, \Omega$

notation: write $H = \{H\}$

$$T = \{T\}$$

since $\{H\} \cup \{T\} = \{H, T\} = \Omega$

Thus $\boxed{P(H) + P(T) = 1}$.

(using axiom (i): $P(\Omega) = 1$ and axiom (iii)a, and $\{H\} \cap \{T\} = \phi$.)

'Fair' means $\boxed{P(H) = P(T)}$

Thus $P(H) = P(T) = \frac{1}{2}$.

In general, if $A \subseteq \Omega$, then $A \cup A^c = \Omega$ and $A \cap A^c = \phi$, so

$$P(A) + P(A^c) = 1$$

Therefore

$$\mathbb{P}(A) \leq 1 \text{ for any } A \subseteq \Omega.$$

and thus

$$0 \leq \mathbb{P}(A) \leq 1$$

Evidently $\mathbb{P} : \underbrace{\mathbb{P}(\Omega)}_{\text{domain may actually be smaller.}} \to [0, 1]$

Letting $A = \Omega$, we see

$$
\begin{aligned}
1 &= P(\Omega) \\
&= P(\Omega \cup \Omega^c) \\
&= P(\Omega \cup \phi) \\
&= P(\Omega) + P(\phi) \\
&= 1 + P(\phi).
\end{aligned}
$$

$\therefore \; P(\phi) = 0.$

Back to example $\Omega = \{H, T\}$

$$\underline{\text{events}} \xrightarrow{P} \underline{\text{Probabilities}}$$

$$\Omega \longrightarrow 1$$

$$H \longrightarrow \tfrac{1}{2}$$

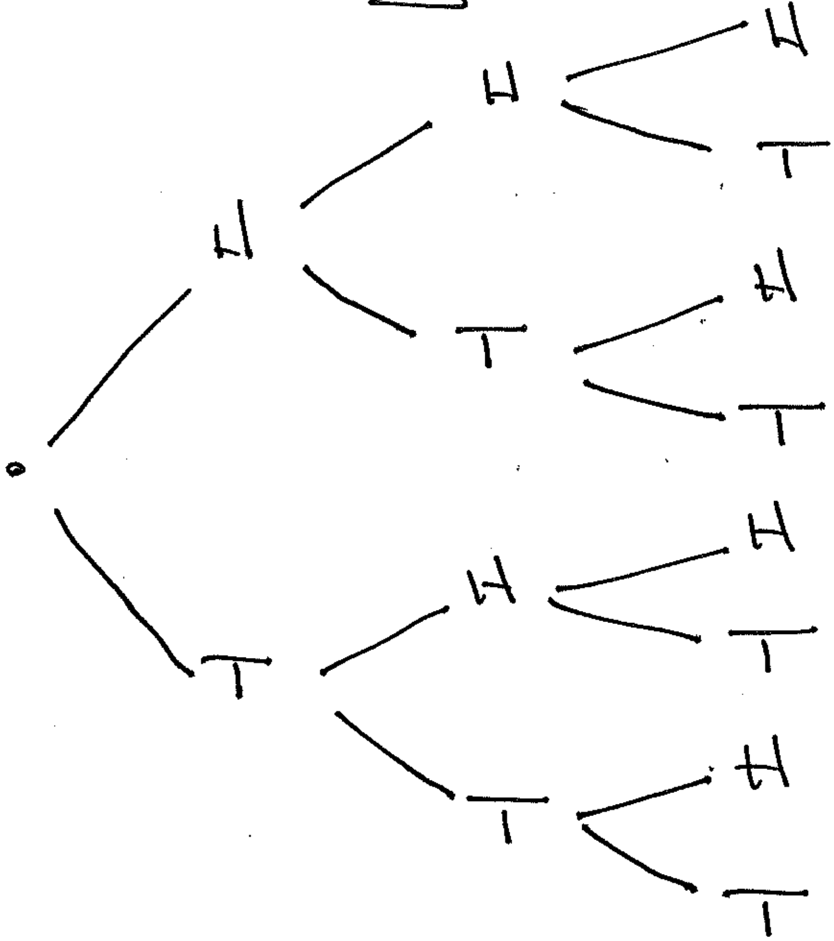
$$T \longrightarrow \tfrac{1}{2}$$

$$\phi \longrightarrow 0$$

<u>EX</u> flip 3 fair coins

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

tree diagram:



outcomes

HHH

HHT

HTH

HTT

THH

THT

TTH

TTT

coin ① ② ③

Fair means all outcomes are equally likely, so each has Prob. $\frac{1}{8}$.

let $A = \{$exactly 1 head$\}$

$= \{HTT, THT, TTH\}$

Since

$$A = \{HTT\} \cup \{THT\} \cup \{TTH\}$$

we have

$$P(A) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \boxed{\frac{3}{8}}$$

Discrete Probability Law

If Ω is finite, then $P(\cdot)$ is specified by the probabilities of singletons

$$P(\omega) = P(\{\omega\}) \text{ for all } \omega \in \Omega.$$

so if

$$A = \{\omega_1, \omega_2, \ldots, \omega_n\},$$

then

$$P(A) = P(\omega_1) + P(\omega_2) + \cdots + P(\omega_n).$$

Special case!

<u>Discrete Uniform Probability law</u>

If Ω is finite, and all outcomes are equally likely, so

$$P(A) = \frac{1}{|\Omega|} + \frac{1}{|\Omega|} + \cdots + \frac{1}{|\Omega|} = \frac{|A|}{|\Omega|}$$

Ex. two 6-sided dice are thrown. The dice are fair so all outcomes are eq. likely

sum= 2	3	4	5	6	7	
11	12	13	14	15	16	8
21	22	23	24	25	26	9
31	32	33	34	35	36	10
41	42	43	44	45	46	11
51	52	53	54	55	56	12
61	62	63	64	65	66	

$$P(\text{sum} = 6) = \frac{5}{36}$$