CSE 107 12-5-24

Final exam: Tue 11/10/24 12-2 PM

SETs: Due SUN. 11/8/24 11:59PM

⋮

<u>Defn</u>

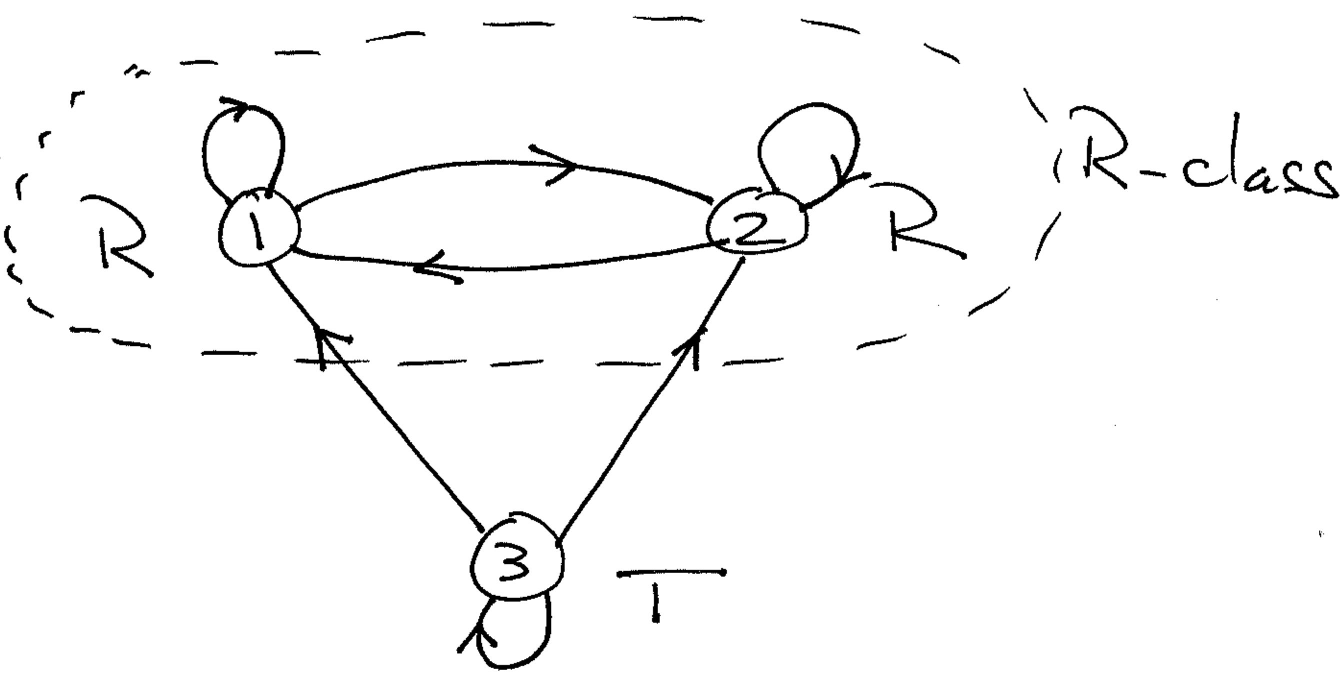
A state i is called <u>Transient</u> iff it is not recurrent. Equivalently there exists $j \in S$ such that

$$j \in A(i) \quad \text{and} \quad i \notin A(j)$$

note:

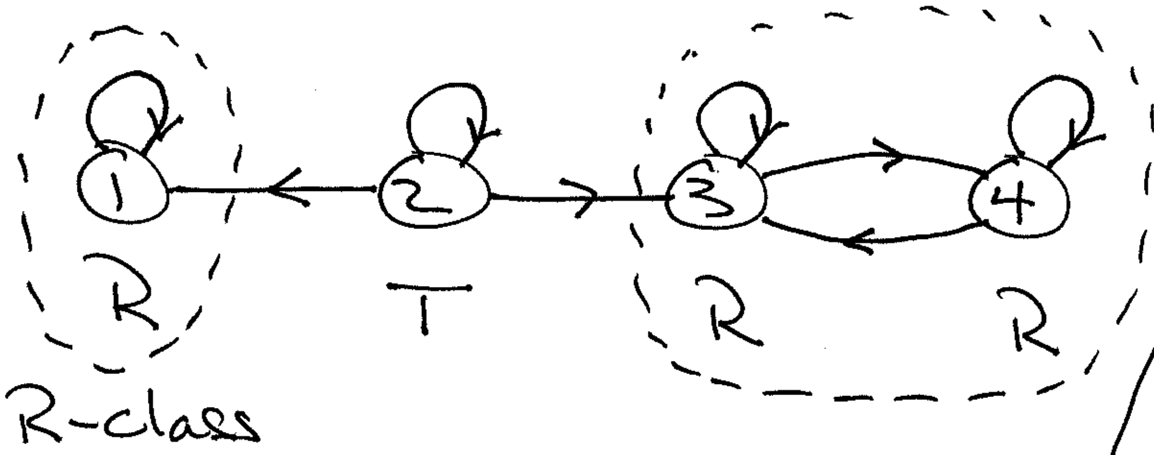
- A transient state will be visited only finitely many times.
- transience and recurrence are determined only by arcs in the transition diagram

Ex.



$A(1) = A(2) = \{1, 2\}$
$A(3) = \{1, 2, 3\}$

Ex



R-class

$A(1)=\{1\}$
$A(2)=\{1,2,3,4\}$
$A(3)=A(4)=\{3,4\}$

note: for any $i,j \in S$

$$j \in A(i) \Rightarrow A(j) \subseteq A(i)$$

and

$$i \in A(j) \Rightarrow A(i) \subseteq A(j)$$

Thus if i is recurrent, then

$$\left.\begin{array}{l} j \in A(i) \Rightarrow A(j) \subseteq A(i) \\ \quad\Updownarrow \\ i \in A(j) \Rightarrow A(i) \subseteq A(j) \end{array}\right\} \Rightarrow A(i)=A(j)$$

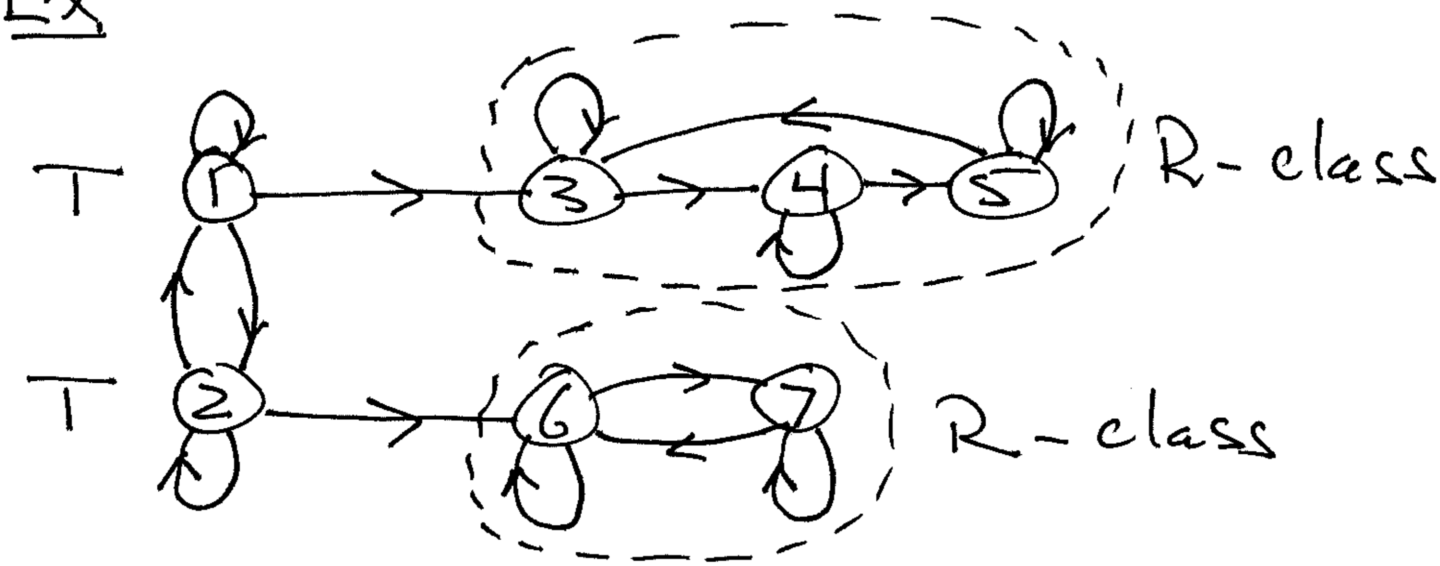
<u>Defn</u>

if i is a recurrent state,

then $A(i) \subseteq S$ is called a

<u>recurrent class</u>.

<u>Ex</u>



$A(1) = A(2) = \{1, 2, 3, 4, 5, 6, 7\}$

$A(3) = A(4) = A(5) = \{3, 4, 5\}$

$A(6) = A(7) = \{6, 7\}$

<u>Exercise</u> (Problem # 6 on P. 381)

Let i be a transient state.
Prove there exists a recurrent
state j such that

$$T \quad (i) \dashrightarrow (j) \quad R$$

i.e. $j \in A(i)$

Problem

Exercise (see #6 on p. 381)

Let i be a transient state. Prove there exists a recurrent state j such that

$$T \ (i) \dashrightarrow (j) \ R$$

i.e.

$$j \in A(i)$$

Theorem: Markov Chain Decomposition

- A Markov chain can be decomposed into one or more recurrent classes, and possibly some transient states.

- A recurrent state is accessible from all states in its class, but not from recurrent states in other classes.

- A transient state is not accessible from any recurrent state

- At least one recurrent state is accessible from a given transient state.

Theorem

(a) when a markov chain enters a recurrent class, it stays within that class, and all states in the class will be visited infinitely often

(b) If the initial state of a Markov chain is transient, then its trajectory consists of a finite sequence of transient states, followed by an infinite sequence of recurrent states belonging to the same class.

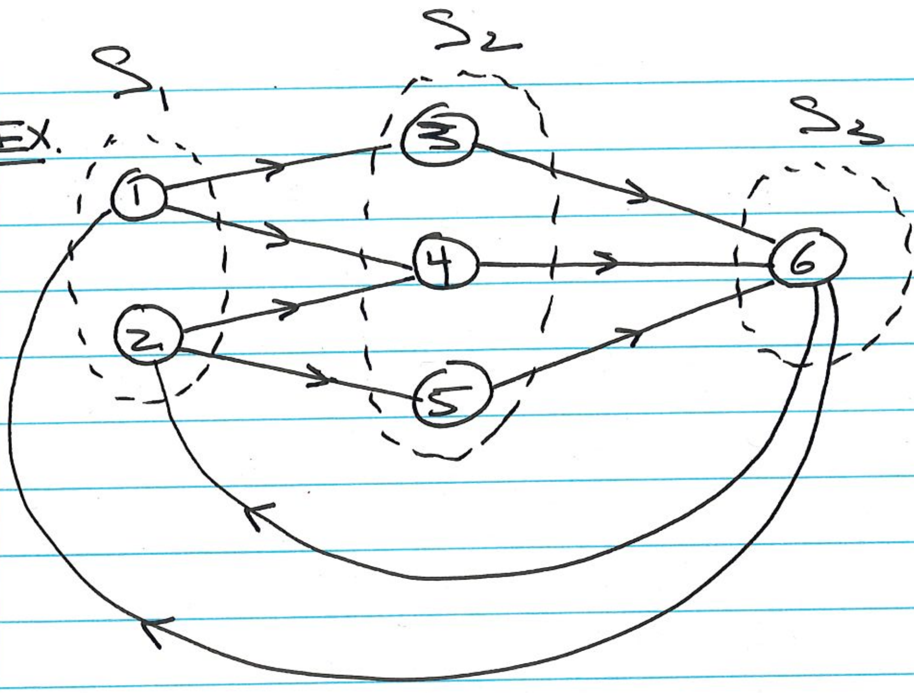
Defn

A recurrent class is called <u>Periodic</u> if its states can be grouped into $d \geq 2$ disjoint subsets $S_1, \ldots, S_d$ such that all transitions from S_k lead to S_{k+1} $(1 \leq k < d)$ and S_d leads to S_1.

$$\begin{tikzcd} S_1 \arrow[r] & S_2 \arrow[r] & S_3 \arrow[r] & \cdots \arrow[r] & S_{d-1} \arrow[r] & S_d \arrow[lllll, bend left] \end{tikzcd}$$

we call d the <u>period</u> of the class

Ex.



Period = 3

observe that in this example

$$X_n \in S_1 \quad \text{iff} \quad n \equiv 0 \pmod{3}$$
$$X_n \in S_2 \quad \text{iff} \quad n \equiv 1 \pmod{3}$$
$$X_n \in S_3 \quad \text{iff} \quad n \equiv 2 \pmod{3}$$

Defn

A recurrent class is called aperiodic iff it is not periodic. Equivalently, there exists a time $n \geq 0$ such that

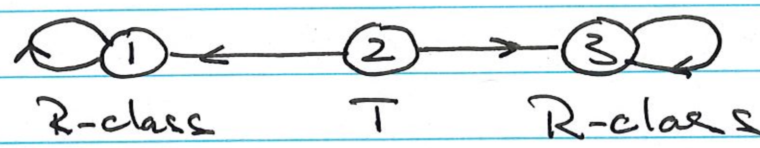
$$r_{ij}(n) > 0 \quad \text{for all} \quad i,j \in R$$

Ex. add a self loop to any state in the previous example.

7.3 steady-state behavior

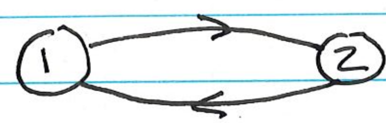
we do not expect the n-step transition probabilities to have limits (not depending on initial state) when there are either

- multiple recurrent classes



or

- periodic recurrent classes



we will write

$$\lim_{n\to\infty} r_{ij}(n) = \pi_j \quad (\text{for all } i)$$

when the limit exists, and we call π_j the <u>steady-state probability</u> of j.

In the case where these limits exist (independent of i) we interpret

$$\pi_j \approx P(X_n = j) \quad \text{(for } n \text{ large)}$$

When do the π_j exist?

<u>Theorem</u> (steady-state convergence) Consider a Markov chain with a single recurrent class, which is aperiodic. Then for each $j \in S$:

$$\lim_{n\to\infty} r_{ij}(n) = \pi_j \quad \text{(for all } i\text{)}$$

exists. Furthermore, π_j are solutions to the linear system

(1) $$\sum_{k=1}^{m} \pi_k p_{kj} = \pi_j \qquad (j = 1, 2, \ldots, m)$$

(2) $$\sum_{k=1}^{m} \pi_k = 1$$

Also $\pi_j = 0$ if j is transient, and $\pi_j > 0$ if j is recurrent.

Equations (1) are called the <u>balance equations</u>, and follow directly from the Chapman-Kolmogorov equations

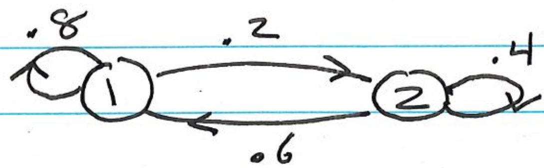
$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1) p_{kj}$$

By taking limits $r_{ij}(n) \to \pi_j$ and $r_{ik}(n-1) \to \pi_k$ as $n \to \infty$.

Equations (2) are callet the <u>normalization equations</u>, and follow from the fact that

$$\sum_{j=1}^{m} P(X_n = j) = 1 \quad (\text{any } n \geq 0)$$

<u>Ex</u>



$$P_{11} = .8 \quad , \quad P_{12} = .2$$
$$P_{21} = .6 \quad , \quad P_{22} = .4$$

$$(1) \quad \begin{cases} \pi_1 P_{11} + \pi_2 P_{21} = \pi_1 \\ \pi_1 P_{12} + \pi_2 P_{22} = \pi_2 \end{cases}$$

$$(2) \quad \pi_1 + \pi_2 = 1$$

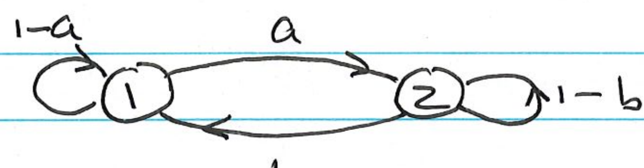
$$\begin{cases} (.8)\pi_1 + (.6)\pi_2 = \pi_1 \iff (.6)\pi_2 = (.2)\pi_1 \\ (.2)\pi_1 + (.4)\pi_2 = \pi_2 \iff (.2)\pi_1 = (.6)\pi_2 \\ \pi_1 + \pi_2 = 1 \end{cases}$$

The 1st two give $\pi_1 = 3\pi_2$. The 3rd yields

$$3\pi_2 + \pi_2 = 1 \rightarrow 4\pi_2 = 1 \rightarrow \boxed{\pi_2 = \frac{1}{4}}$$

hence $\boxed{\pi_1 = \frac{3}{4}}$

Ex.



$$\pi_1(1-a) + \pi_2 b = \pi_1$$
$$\pi_1 a + \pi_2(1-b) = \pi_2$$

$$\pi_2 b = \pi_1 a$$

$$\therefore \quad \pi_1 = \frac{b}{a}\pi_2$$

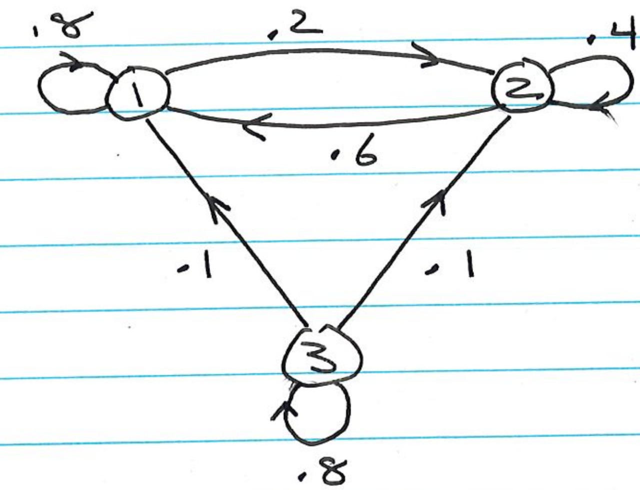
$$\therefore \quad \frac{b}{a}\pi_2 + \pi_2 = 1$$

$$\therefore \quad \pi_2 = \frac{1}{\frac{b}{a}+1} \rightarrow \boxed{\pi_2 = \frac{a}{a+b}}$$

$$\therefore \quad \boxed{\pi_1 = \frac{b}{a+b}}$$

L13

Ex.



$$\begin{cases} \pi_1 P_{11} + \pi_2 P_{21} + \pi_3 P_{31} = \pi_1 \\ \pi_1 P_{12} + \pi_2 P_{22} + \pi_3 P_{32} = \pi_2 \\ \pi_1 P_{13} + \pi_2 P_{23} + \pi_3 P_{33} = \pi_3 \end{cases}$$

$$\begin{cases} (.8)\pi_1 + (.6)\pi_2 + (.1)\pi_3 = \pi_1 \\ (.2)\pi_1 + (.4)\pi_2 + (.1)\pi_3 = \pi_2 \end{cases}$$

$$\begin{cases} -2\pi_1 + 6\pi_2 + \pi_3 = 0 \\ 2\pi_1 - 6\pi_2 + \pi_3 = 0 \end{cases} \rightarrow 2\pi_3 = 0 \rightarrow \boxed{\pi_3 = 0}$$

$$\rightarrow \pi_1 = 3\pi_2$$

$$3\pi_2 + \pi_2 = 1 \rightarrow \boxed{\pi_2 = \frac{1}{4}} \quad \therefore \boxed{\pi_1 = \frac{3}{4}}$$

See examples 7.6 and 7.7 on Pages 355 and 356, resp.

we should have anticipated this since we know state 3 is transient