

CSS 107 12-3-24

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Final Exam: Tue 12/10, 12:00-2:00 PM

SETS: Due Sun 12/8 at 11:59 PM

if response rate $\geq 85\%$, I

will add 0.5% to all
overall scores.

7.1 Discrete Markov Chains

let S be a set of states

$$S = \{1, 2, 3, \dots, m\}$$

let $X_0, X_1, X_2, \dots$ be a seq.
of r.v.s taking values in S .
consider

X_n = state of process at
time n .

Define (Transition Probabilities)

$$p_{ij} = P(X_{n+1} = j \mid X_n = i)$$

Markov Property

$$P(X_{n+1}=j | X_n=i, X_{n-1}=i_{n-1}, \dots, X_0=i_0)$$

$$= P(X_{n+1}=j | X_n=i)$$

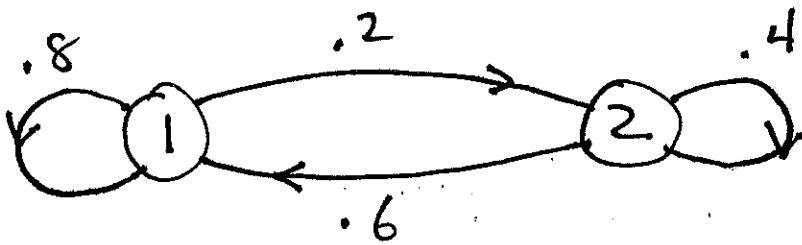
Defn

A Markov chain model consists of

S , P_{ij} and the Markov Property.

Ex. $m=2$, $S=\{1, 2\}$

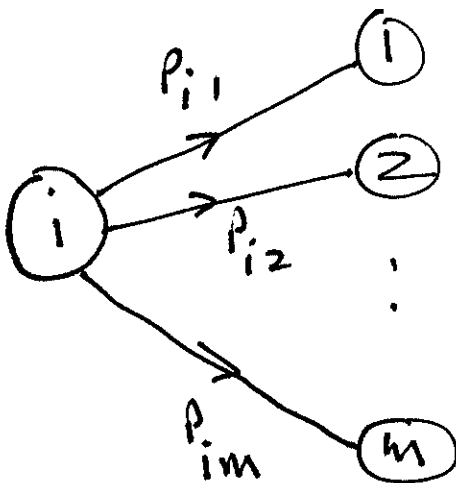
$$P_{11} = .8, P_{12} = .2, P_{21} = .6, P_{22} = .4$$



state
transition
diag-ram

note. Transition Probabilities satisfy

$$\sum_{j=1}^m P_{ij} = 1 \quad (\text{for all } i \in S)$$



Suppose $X_0 = 1$. Find $P(X_2 = 2)$

By total Prob. Then

$$\begin{aligned} P(X_2 = 2) &= P(X_2 = 2 | X_1 = 1) \cdot P(X_1 = 1) \\ &\quad + P(X_2 = 2 | X_1 = 2) \cdot P(X_1 = 2) \\ &= p_{12} \cdot P(X_1 = 1) + p_{22} \cdot P(X_1 = 2) \end{aligned}$$

$$\begin{aligned} P(X_1 = 1) &= P(X_1 = 1 | X_0 = 1) \cdot \overbrace{P(X_0 = 1)}^1 \\ &\quad + P(X_1 = 1 | X_0 = 2) \cdot \overbrace{P(X_0 = 2)}^0 \\ &= p_{11} \end{aligned}$$

$$P(X_1=2) = P(X_1=2|X_0=1) \cdot \overbrace{P(X_0=1)}^1 + 0$$

$$= P_{12}$$

$$\therefore P(X_2=2) = P_{12} \cdot P_{11} + P_{22} \cdot P_{12}$$

$$= P_{11} \cdot P_{12} + P_{12} \cdot P_{22}$$

$$= (.8)(.2) + (.2)(.4) = .16 + .08 = \boxed{.24}$$

Similarly

$$P(X_2=1) = P_{11} \cdot P_{11} + P_{12} P_{21}$$

$$= P_{11}^2 + P_{12} P_{21} = \dots = \boxed{.76}$$

In the same manner, if $X_0 = 2$,
then

$$P(X_2 = 1) = P_{21}P_{11} + P_{22}P_{21} = \dots = \boxed{.72}$$

and

$$P(X_2 = 2) = P_{21}P_{12} + P_{22}P_{22} = \dots = \boxed{.28}$$

To summarize, define transition
Probability matrix

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \begin{pmatrix} .8 & .2 \\ .6 & .4 \end{pmatrix}$$

observe

$$R^2 = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$$

$$= \begin{pmatrix} P_{11}^2 + P_{12}P_{21} & P_{11}P_{12} + P_{12}P_{22} \\ P_{21}P_{11} + P_{22}P_{21} & P_{21}P_{12} + P_{22}^2 \end{pmatrix} = \begin{pmatrix} .76 & .24 \\ .72 & .28 \end{pmatrix}$$

Defn

A Probability matrix is a matrix with pos. real entries in which rows add to 1.

Exercise Show that the product of Probability matrices is a Probability matrix.

Thus the i^{th} element in R^2 is the Prob. of going from (i) to (i) in 2 steps

Defn

The n -step transition probabilities are

$$r_{ij}(n) = P(X_n = j | X_0 = i)$$

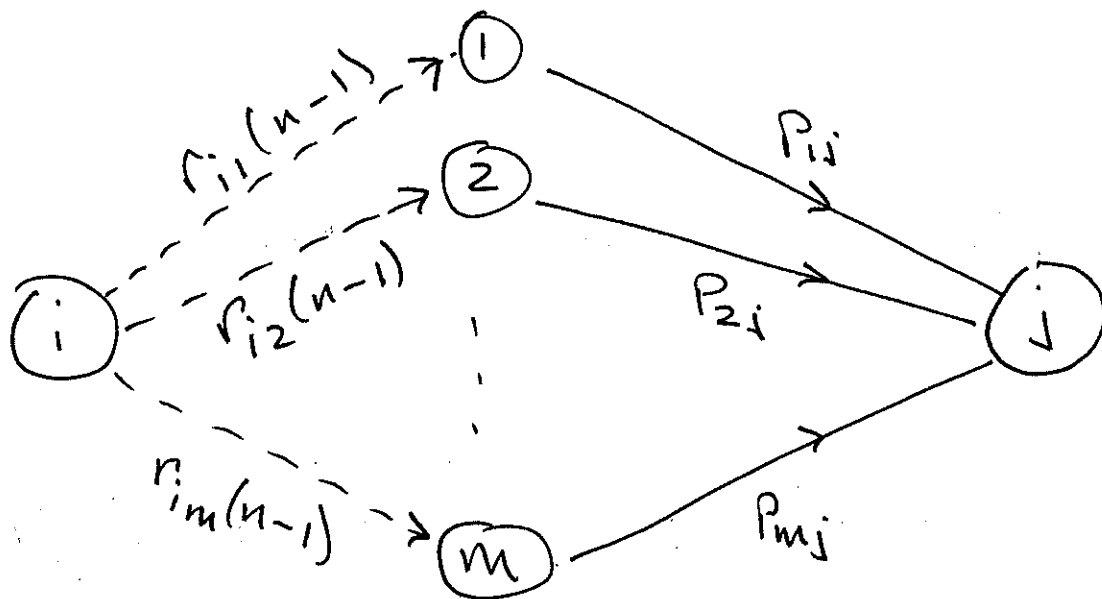
$$= P(X_{n+n_0} = j | X_{n_0} = i)$$

for any $i, j \in S$ and $n \geq 0$.

note $r_{ij}(1) = p_{ij}$.

Chapman-Kolmogorov Equations

$$r_{ij}(n) = \sum_{k=1}^m r_{ik}(n-1) \cdot p_{kj}$$



Proof. apply total Prob. thm, or see text.

matrix form of C.K. eqns.

$$\left(r_{ij}(n) \right)_{i,j \in S} = \overset{n-1}{\underset{\sim}{R}} \cdot \overset{n}{\underset{\sim}{R}} = \overset{n}{\underset{\sim}{R}}$$

Ex. Previous

$$\overset{3}{\underset{\sim}{R}} = \dots = \begin{pmatrix} .752 & .248 \\ .744 & .256 \end{pmatrix}$$

$$\text{so } r_{2,1}(3) = P(X_3=1 | X_0=2) = .744$$

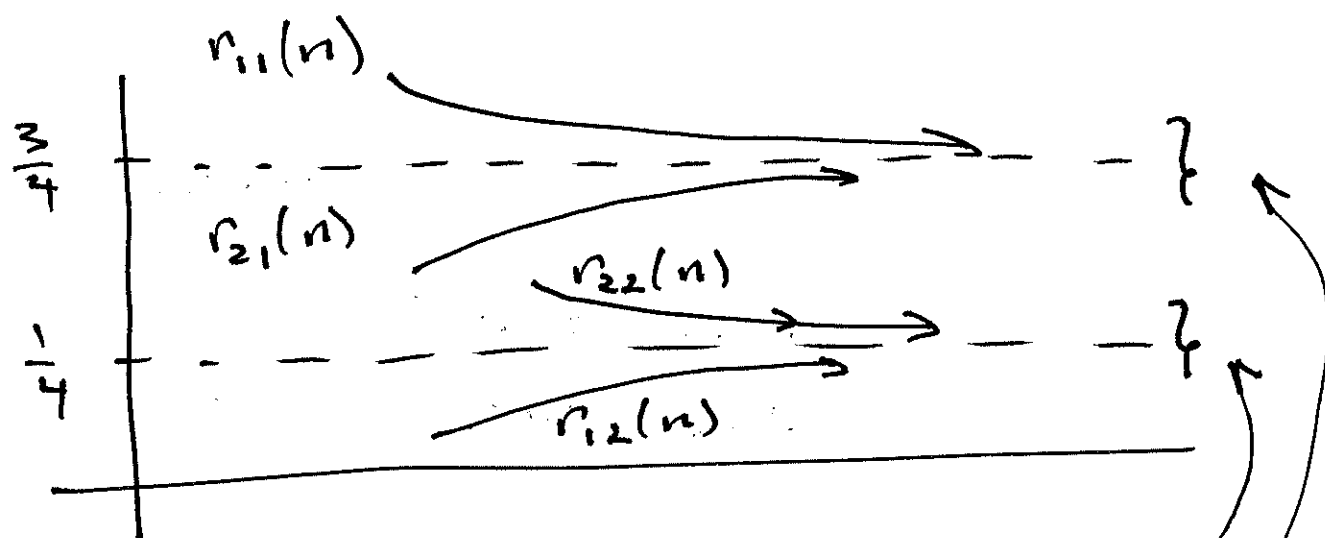
we can show (using L.A.)

$$\lim_{n \rightarrow \infty} \overset{n}{\underset{\sim}{R}} = \begin{pmatrix} .75 & .25 \\ .75 & .25 \end{pmatrix}$$

i.e.

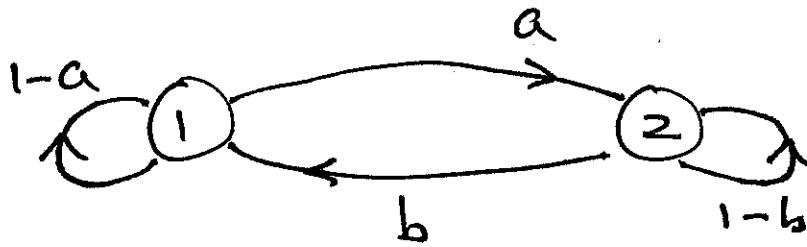
$$r_{11}(n) \rightarrow \frac{3}{4}^+, \quad r_{12}(n) \rightarrow \frac{1}{4}^-$$

$$r_{21}(n) \rightarrow \frac{3}{4}^-, \quad r_{22}(n) \rightarrow \frac{1}{4}^+$$



Steady state Probabilities

Ex.



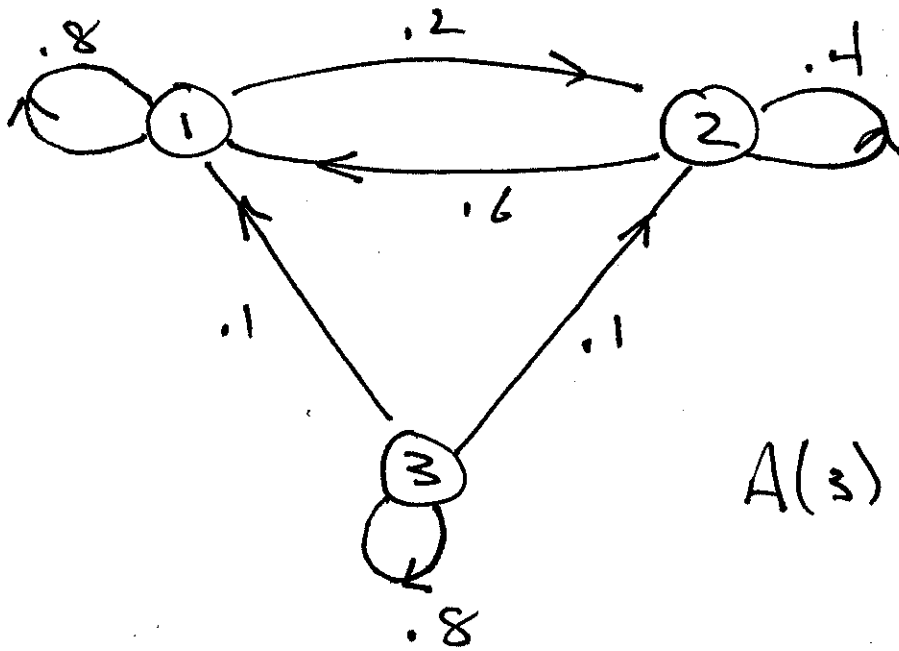
$$R = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$

one can show

$$\lim_{n \rightarrow \infty} R^n = \begin{pmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{pmatrix}$$

Ex.

$$\Delta(1) = \Delta(2) = \{1, 2\}$$



$$\Delta(3) = \{1, 2, 3\}$$

$$R = \begin{pmatrix} .8 & .2 & 0 \\ .6 & .4 & 0 \\ .1 & .1 & .8 \end{pmatrix}$$

one can show

$$\lim_{n \rightarrow \infty} R^n = \begin{pmatrix} .75 & .25 & 0 \\ .75 & .25 & 0 \\ .75 & .25 & 0 \end{pmatrix}$$

7.2 classification of states

Defn

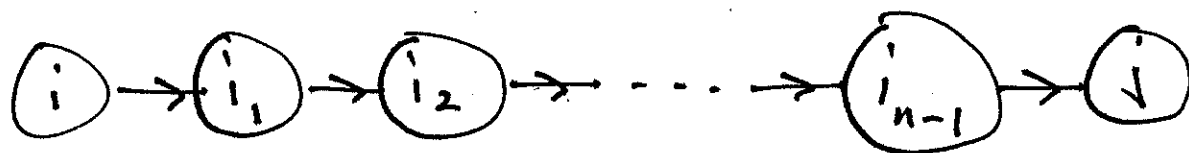
we say state j is accessible from state i (or that j is reachable from i) iff

$$r_{ij}(n) > 0 \text{ for some } n \geq 1$$

Equivalently, there exists

$$i, i_1, i_2, \dots, i_{n-1}, j$$

such that the transitions



all have Positive Probability

notation: $(i) \dashrightarrow (j)$

means: i is accessible from j

Defn

Let $A(i)$ denote the set of states accessible from i

$$A(i) = \{j \in S \mid (i) \dashrightarrow (j)\}$$

Defn

A state i is called recurrent iff
for every $j \in A(i)$, we also have
 $i \in A(j)$

Equivalently:

$$j \in A(i) \text{ iff } i \in A(j)$$

i.e.

$$(i) \rightarrow (j) \text{ iff } (j) \rightarrow (i)$$

Thus if we depart a recurrent state, there is a pos. prob. that we return. So after long enough time, we will return.

Thus a recurrent state, if it is visited at all, is visited infinitely often.