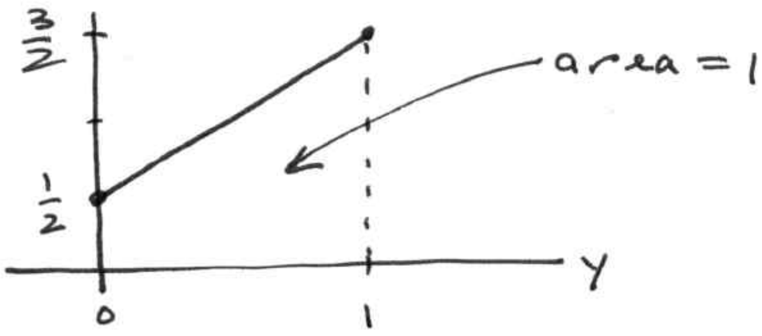
cse 107 11-8-24

Supplemental Lecture

Ex

let X, Y be jointly continuous with

$$f_Y(y) = \begin{cases} y + \frac{1}{2} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$



and

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{x+y}{y+\frac{1}{2}} & 0 \le x \le 1, \ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

find $E[X]$.

<u>Solution</u>

$$E[X|Y=y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\,dx$$

$$= \int_0^1 x\left(\frac{x+y}{y+\frac{1}{2}}\right)dx = \frac{1}{y+\frac{1}{2}}\int_0^1 (x^2+xy)\,dx$$

$$= \frac{1}{y+\frac{1}{2}}\left(\frac{1}{3}x^3+\frac{1}{2}x^2y\right)\Bigg|_0^1 = \frac{\frac{1}{2}y+\frac{1}{3}}{y+\frac{1}{2}} \qquad (0 \le y \le 1)$$

so

$$E[X] = \int_{-\infty}^{\infty} E[X|Y=y]\cdot f_Y(y)\,dy$$

$$= \int_0^1 \left(\frac{\frac{1}{2}y+\frac{1}{3}}{\cancel{y+\frac{1}{2}}}\right)\cancel{(y+\tfrac{1}{2})}\,dy$$

$$= \frac{1}{2}\cdot\frac{1}{2}y^2+\frac{1}{3}y\Bigg|_0^1 = \frac{1}{4}+\frac{1}{3} = \boxed{\frac{7}{12}}$$

another way:

$$f_{X,Y}(x,y) = f_{X|Y}(x|y) \cdot f_Y(y)$$

$$= \begin{cases} x+y & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

so

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= \int_0^1 (x+y)\,dy = xy + \tfrac{1}{2}y^2 \Big|_0^1 = x + \tfrac{1}{2}$$

$$= \begin{cases} x+\tfrac{1}{2} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= \int_0^1 x\left(x+\tfrac{1}{2}\right)dx = \int_0^1 \left(x^2+\tfrac{1}{2}x\right)dx$$

$$= \tfrac{1}{3}x^3 + \tfrac{1}{2}\cdot\tfrac{1}{2}x^2 \Big|_0^1$$

$$= \tfrac{1}{3} + \tfrac{1}{4} = \boxed{\tfrac{7}{12}}$$

note: $E[X]$ is a number, while

$$E[X \mid Y=y] = \begin{cases} \dfrac{\frac{1}{2}y + \frac{1}{3}}{y+\frac{1}{2}} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

we can write

$$g(y) = E[X \mid Y=y]$$

we can compose $g(y)$ with Y

to get

$$g(Y) = E[X|Y]$$

our first computation showed

$$E[E[X|Y]] = \frac{7}{12}$$

we'll see in chapter 4 that:

$$\boxed{E[E[X|Y]] = E[X]}$$

another way of writing (2)

called: <u>law of iterated expectations</u>

Independence

let X, Y be jointly continuous

Defn

we say X, Y are independent iff

- $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$

for all $x \in \text{range}(X)$, $y \in \text{range}(Y)$.

Equivalently:

- $f_{X|Y}(x|y) = f_X(y)$ $\left(\begin{array}{c}\text{for } y \text{ s.t.} \\ f_Y(y) > 0\end{array}\right)$

or

- $f_{Y|X}(y|x) = f_Y(y)$ $\left(\begin{array}{c}\text{for } x \text{ s.t.} \\ f_X(x) > 0\end{array}\right)$

If X, Y are independent, then any events definable in terms of X and Y are also independent

$$\{X \in S\} \text{ and } \{Y \in T\}$$

In Particular

$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$$

$$= \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(s,t)\, ds\, dt$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f_X(s) \cdot f_Y(t)\, ds\, dt$$

$$= \int_{-\infty}^{x} f_X(s)\, ds \cdot \int_{-\infty}^{y} f_Y(t)\, dt$$

$$= P(X \leq x) \cdot P(Y \leq y) = F_X(x) \cdot F_Y(y)$$

<u>Fact</u>: the converse is also true, i.e. if $F_{X,Y} = F_X \cdot F_Y$ the $f_{X,Y} = f_X \cdot f_Y$, and X, Y are independent.

As in discrete case we have for r.v.s X, Y independent that

(1) $E[X \cdot Y] = E[X] \cdot E[Y]$

(2) $E[g(X) \cdot h(Y)] = E[g(X)] \cdot E[h(Y)]$

(3) $Var(X+Y) = Var(X) + Var(Y)$

<u>Exercise</u>

- Prove (1) in continuous case

- Prove that if X, Y independent, then $g(X), h(Y)$ are also independent.

- Prove (3) in continuous case. Mimic proof from discrete case.

<u>Hint</u>: adapt the proof in Problem #44, ch. 2, p. 134

3.6 Bayes Rule

Recall Bayes rule!

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

If $A_1, A_2, \ldots, A_n$ form a partition of Ω, we use total probability thm to get

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i)$$

$$= \sum_{i=1}^{n} P(B|A_i) \cdot P(A_i)$$

Hence

$$P(A_k|B) = \frac{P(B|A_k) \cdot P(A_k)}{\sum_{i=1}^{n} P(B|A_i) \cdot P(A_i)}$$

for any k in range $1 \le k \le n$.

If X, Y are discrete r.v.s then this becomes

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x) \cdot p_X(x)}{\sum_t p_{Y|X}(y|t) \cdot p_X(t)}$$

In the continuous case this is

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) \cdot f_X(x)}{\int_{-\infty}^{\infty} f_{Y|X}(y|t) f_X(t)\, dt}$$

Ex.

Let Y be normal with $\sigma = 1$, and mean X, another r.v. Suppose X is continuous uniform on $[1, 3]$.

a.) find $f_Y(y)$

Solution

we are given:

$$f_X(x) = \begin{cases} \frac{1}{2} & 1 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{\sqrt{2\pi}} \cdot e^{-(y-x)^2/2} & \begin{pmatrix} 1 \le x \le 3 \\ y \in \mathbb{R} \end{pmatrix} \\ 0 & \text{otherwise} \end{cases}$$

so

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_X(x)\,dx$$

$$= \frac{1}{2}\int_1^3 \frac{1}{\sqrt{2\pi}} e^{-(x-y)^2/2}\,dx$$

sub: $u = x - y$
$du = dx$

$$= \frac{1}{2}\int_{1-y}^{3-y} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\,du$$

$$= \frac{1}{2}\left(\Phi(3-y) - \Phi(1-y) \right)$$

b). find $f_{X|Y}(x|y)$.

use Baye's rule.

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) \cdot f_X(x)}{f_Y(y)}$$

$$= \begin{cases} \dfrac{\frac{1}{\sqrt{2\pi}} \cdot e^{-(y-x)^2/2}}{\Phi(3-y) - \Phi(1-y)} & \left(\begin{array}{c} 1 \le x \le 3 \\ y \in \mathbb{R} \end{array}\right) \\ 0 & \text{otherwise} \end{cases}$$

c.) suppose we sample Y and get $y = 3$. What is the Prob. that $X \le 2$?

d.) find $E[Y]$.