CSE 107 11-7-24

Ex.

The time T until a new light bulb burns out is an exponential r.v. with parameter λ. Alice turns on the light at time $t=0$. She returns t_0 hours later, finding light still on. i.e. the event

$$A = \{T > t_0\}$$

has occurred. note

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Let X be the additional time to failure, i.e. $X = T - t_0$ or $T = X + t_0$. Find $f_{X|A}(x)$.

<u>Solution</u>

first find cond. CDF

$$F_{X|A}(x) = P(X \le x \mid A)$$

$$= \int_{-\infty}^{x} f_{X|A}(s)\,ds$$

then differentiate:

$$\frac{d}{dx} F_{X|A}(x) = f_{X|A}(x).$$

Recall (11-29-24, p.14)

$$P(T > a) = \begin{cases} e^{-\lambda a} & \text{if } a > 0 \\ 1 & \text{if } a \leq 0 \end{cases}$$

Thus, for $x \geq 0$:

$$P(X > x \mid A) = P(T > t_0 + x \mid T > t_0)$$

$$= \frac{P(T > t_0 + x, T > t_0)}{P(T > t_0)}$$

$$= \frac{P(T > t_0 + x)}{P(T > t_0)}$$

$$= \frac{e^{-\lambda(t_0 + x)}}{e^{-\lambda t_0}} = \frac{\cancel{e^{-\lambda t_0}} \cdot e^{-\lambda x}}{\cancel{e^{-\lambda t_0}}} = e^{-\lambda x}$$

Thus

$$F_{X|A}(x) = P(X \le x \mid A)$$

$$= 1 - P(X > x \mid A)$$

$$= \begin{cases} 1 - e^{-\lambda x} & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Hence

$$f_{X|A}(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$$

so cond. PDF is again exponential with same param. i.e. exponential r.v.s have no memory.

Exercise

Do same thing for the geometric r.v.

$$\overbrace{\underbrace{0\,0\,0\,0\cdots\cdots0}_{m}\;\underbrace{0\,0\cdots\cdots0\,1}_{X}}^{Y}$$

let

$Y = \#$ flips til 1^{st} head

$X = \#$ flips after m^{th} to 1^{st} head

$A = \{Y > m\}$.

show : $P_{X|A}(k) = P_Y(k) \qquad (k = 1, 2, 3, \ldots)$

Recall : $P_Y(k) = p(1-p)^{k-1} \qquad (k = 1, 2, 3, \ldots)$

If events $A_1, A_2, \ldots, A_n$ form a partition of Ω, then the Total Prob. Thm. says

$$P(X \le x) = \sum_{i=1}^{n} P(A_i) \cdot P(X \le x \mid A_i)$$

Thus

$$F_X(x) = \sum_{i=1}^{n} P(A_i) \cdot F_{X|A_i}(x)$$

diff. w.r.t. x to get

$$\boxed{f_X(x) = \sum_{i=1}^{n} P(A_i) \cdot f_{X|A_i}(x)}$$

Defn

If X, Y are jointly continuous r.v.s and $f_Y(y) > 0$ for all $y \in \mathbb{R}$,

the conditional PDF of X given Y is

$$\boxed{f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}}$$

note:

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y)\,dx = \int_{-\infty}^{\infty} \frac{f_{X,Y}(x,y)}{f_Y(y)}\,dx$$

$$= \frac{\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx}{f_Y(y)} = \frac{f_Y(y)}{f_Y(y)} = 1 \quad \checkmark$$

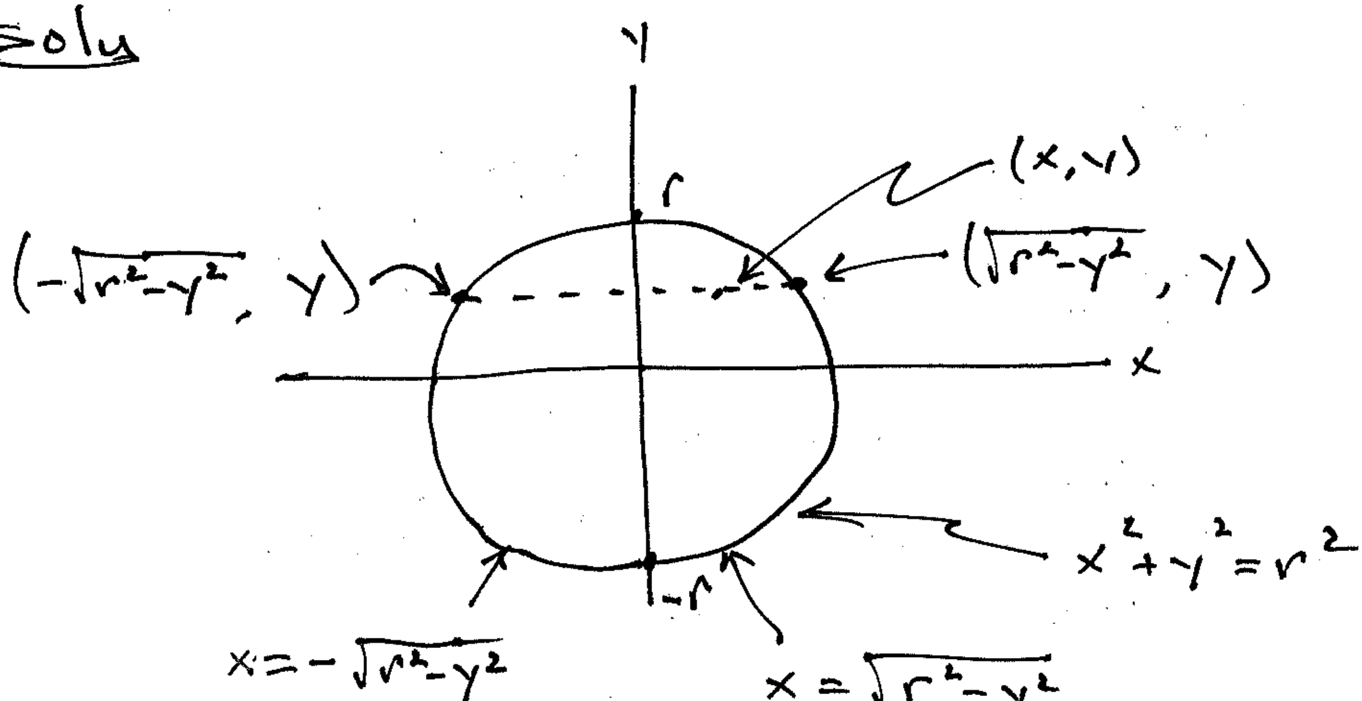
Ex.

Throw a dart at circular target of radius r, centered at $(0,0)$.

Assume target is always hit, and otherwise the landing point (x,y) is uniformly distributed. Let X, Y be the coordinates hit.

Find $f_{X,Y}(x,y)$, $f_Y(y)$ and $f_{X|Y}(x|y)$.

Solu



we're given

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi r^2} & \text{if } x^2+y^2 \le r^2 \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{1}{\pi r^2}\,dx$$

$$= \frac{1}{\pi r^2} \cdot 2\sqrt{r^2-y^2}$$

$$= \begin{cases} \frac{2\sqrt{r^2-y^2}}{\pi r^2} & \text{if } -r \le y \le r \\ 0 & \text{otherwise} \end{cases}$$

Finally

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \begin{cases} \dfrac{1}{2\sqrt{r^2-y^2}} & \text{if } x^2+y^2 \le r \\ 0 & \text{otherwise} \end{cases}$$

The PMF identities all hold for PDFs. for instance

$$f_{X|Y}(x|y) \cdot f_Y(y) = f_{X,Y}(x,y) = f_{Y|X}(y|x) \cdot f_X(x)$$

also multiplication rule:

$$f_{X,Y,Z}(x,y,z) = f_{X|Y,Z}(x|y,z) \cdot f_{Y|Z}(y|z) \cdot f_Z(z)$$

Conditional Expectation

Let X, Y be jointly cont., and $A \subseteq \Omega$ with $P(A) > 0$.

Definitions

- $E[X|A] = \int_{-\infty}^{\infty} x f_{X|A}(x)\,dx$

- $E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx$

expected value rules:

- $E[g(X)|A] = \int_{-\infty}^{\infty} g(x) f_{X|A}(x)\,dx$

- $E[g(X)|Y=y] = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y)\,dx$

total expectation thm

(1) let $A_1, \dots, A_n$ be a partition of Ω and $P(A_i) > 0$ for $i = 1, \dots, n$.

$$E[X] = \sum_{i=1}^{n} P(A_i) \cdot E[X \mid A_i]$$

(2) let Y satisfy $f_Y(y) > 0$ for all $y \in \mathbb{R}$

note events $\{Y = y\}$ form a partition of Ω, where $y \in \text{range}(Y)$.

$$E[X] = \int_{-\infty}^{\infty} f_Y(y) \cdot E[X \mid Y = y] \, dy$$

<u>Proof of (1)</u>

start with total. Prob. thm.

$$f_X(x) = \sum_{i=1}^{n} P(A_i) f_{X|A_i}(x)$$

so

$$\int_{-\infty}^{\infty} x f_X(x)dx = \int_{-\infty}^{\infty} x \cdot \sum_{i=1}^{n} P(A_i) f_{X|A_i}(x)\, dx$$

$$= \sum_{i=1}^{n} P(A_i) \cdot \int_{-\infty}^{\infty} x f_{X|A_i}(x)\, dx$$

Thus

$$E[X] = \sum_{i=1}^{n} P(A_i) \cdot E[X|A_i]$$

$\blacksquare$

Proof of (2)

Start with RHS

$$\int_{-\infty}^{\infty} E[X \mid Y=y] \cdot f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx \right) \cdot f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y) \cdot f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \cdot f_{X,Y}(x,y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} x \cdot \left(\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \right) dx$$

$$= \int_{-\infty}^{\infty} x \, f_X(x) \, dx$$

$$= E[X]$$

$\blacksquare$