CSE 107 11-5-24

3.4 Joint PDF of r.v.s X, Y

Defn
We say r.v.s X, Y are jointly continuous iff there exists a function $f_{X,Y}(x,y)$ such that

$$P((X,Y)\in S) = \iint_S f_{X,Y}(x,y)\,dx\,dy$$

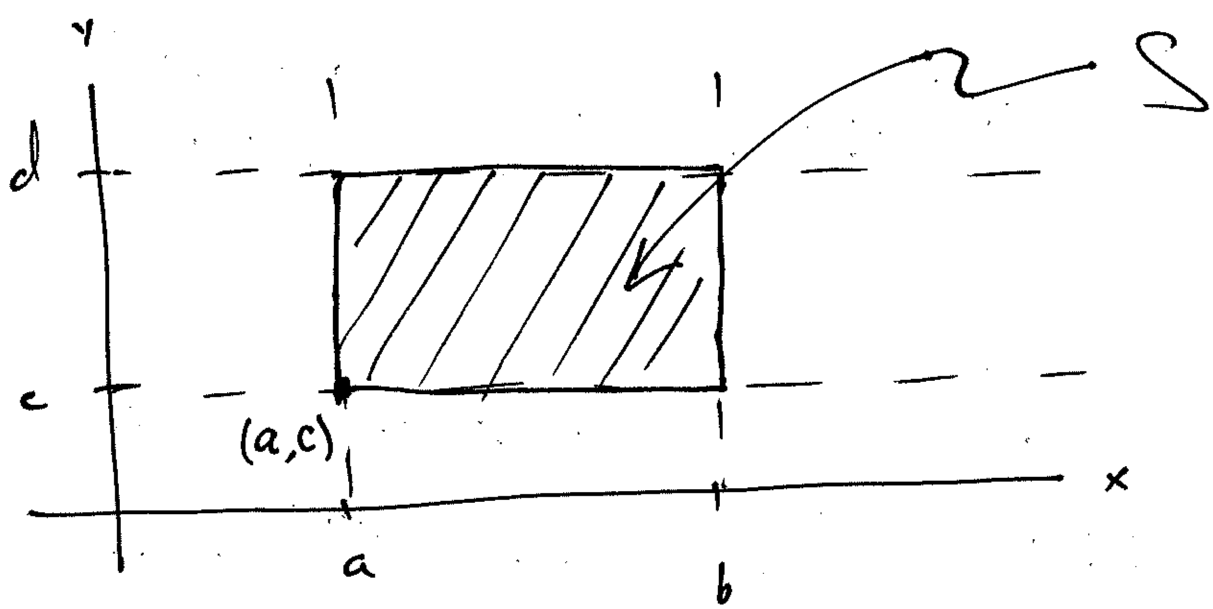
for 'any' subset $S \subseteq \mathbb{R}^2$. We call $f_{X,Y}(x,y)$ the joint PDF of X, Y.

In Particular if

$$S = [a,b] \times [c,d] = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$$

Then

$$P(a \le X \le b,\ c \le Y \le d) = \int_a^b \int_c^d f_{X,Y}(x,y)\,dy\,dx$$



If $S = \mathbb{R}^2$, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = P(\Omega) = 1$$

If $\delta > 0$ is small and $f_{X,Y}(x,y)$ is continuous near (a,c), then

$$P(a \le X \le a+\delta,\ c \le Y \le c+\delta)$$

$$= \int_{a}^{a+\delta}\int_{c}^{c+\delta} f_{X,Y}(x,y)\,dy\,dx$$

$$\approx f_{X,Y}(a,c)\cdot \delta^2$$

<u>note</u>: δ^2 has units of area,

so $f_{X,Y}(a,c)$ is probability per

unit area, i.e. a density.

Recall if $R \subseteq \mathbb{R}$, we have

$$P(X \in R) = \int_R f_X(x)\,dx$$

But also

$$P(X \in R) = P(X \in R, Y \in \mathbb{R})$$

$$= \iint_{R \times \mathbb{R}} f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{R} \left(\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \right) dx$$

Therefore: (marginal PDF)

$$\boxed{f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy}$$

Similarly

$$\boxed{f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx}$$

observe

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} x \left(\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \right) dx$$

$$= \iint_{\mathbb{R}^2} x f_{X,Y}(x,y)\, dy\, dx$$

Similarly

$$E[Y] = \iint_{\mathbb{R}^2} y f_{X,Y}(x,y)\, dx\, dy$$

□

Ex. Joint uniform continuous PDF on $S = [a,b] \times [c,d]$

$$f_{X,Y}(x,y) = \begin{cases} \alpha & \text{if } (x,y) \in S \\ 0 & \text{if } (x,y) \notin S \end{cases}$$

we must have

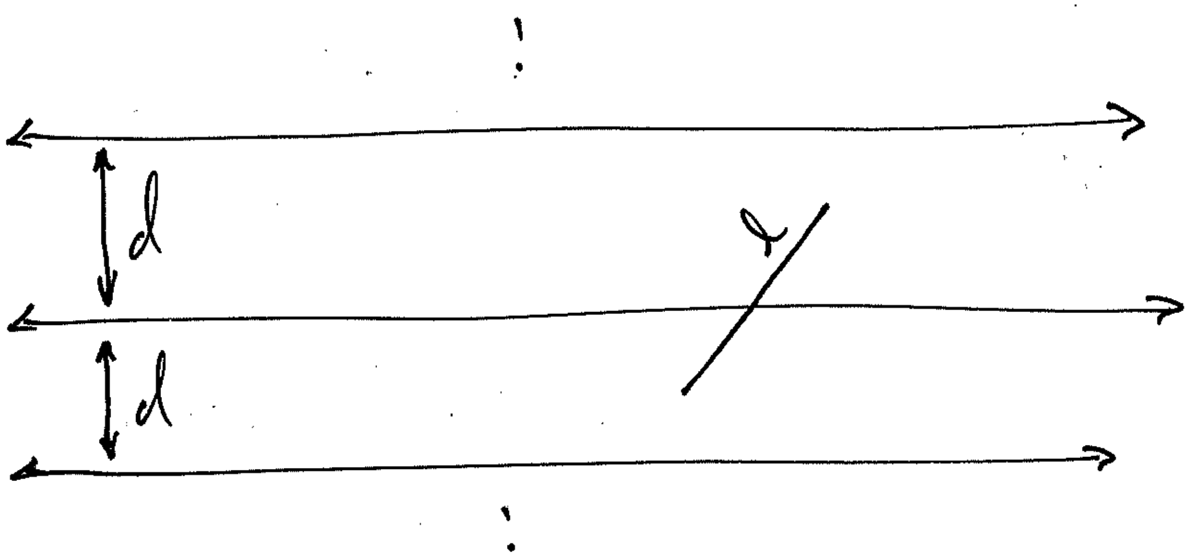
$$1 = \iint_{\mathbb{R}^2} f_{X,Y}(x,y)\,dx\,dy = \iint_S \alpha\,dx\,dy$$

$$= \alpha \cdot \text{area}(S)$$

$$\therefore \quad \alpha = \frac{1}{\text{area}(S)} = \frac{1}{(b-a)(c-d)}$$

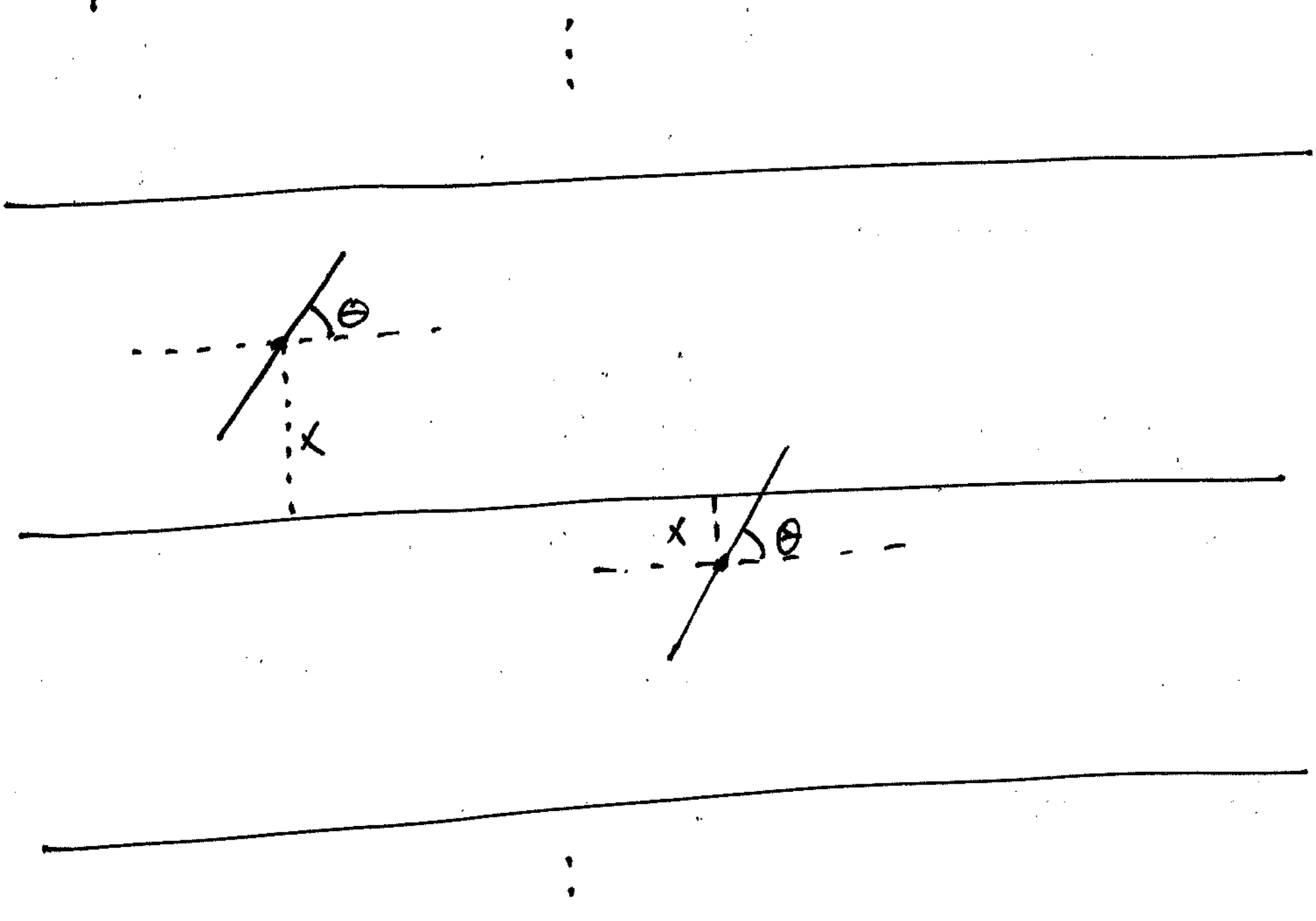
Ex. Buffon's Needle

A surface is ruled with Parallel lines distance d apart



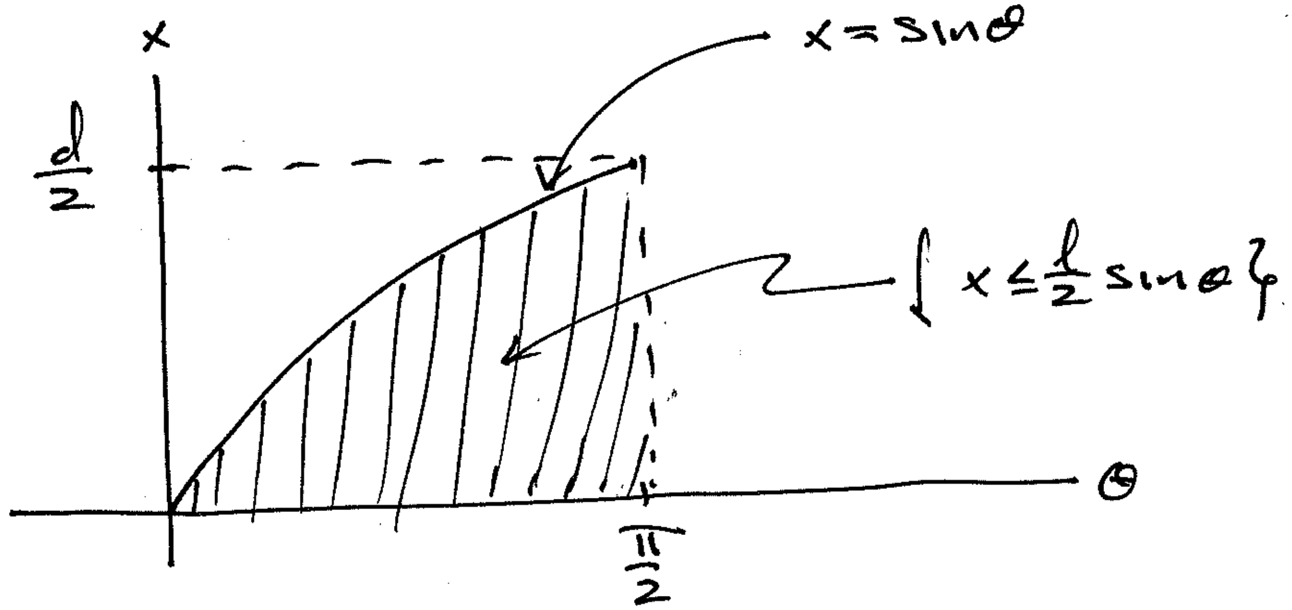
A needle of length l is tossed on the surface. What is the Probability that the needle intersects one of the lines. (assume $l<d$)

Let X be vertical distance from midpoint of needle to the nearest line, let Θ be the acute angle made by the needle and the horizontal

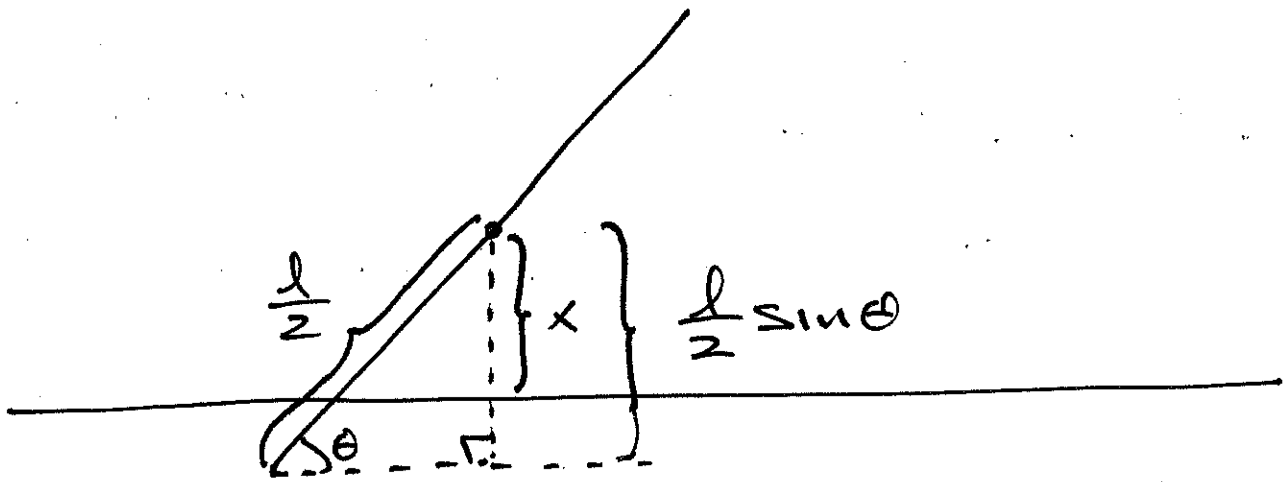


we model (X, Θ) with a joint uniform PDF on $[0, \frac{d}{2}] \times [0, \frac{\pi}{2}]$, with area $= \frac{d\pi}{4}$. Thus

$$f_{X,\Theta}(x,\theta) = \begin{cases} \frac{4}{\pi d} & \text{if } 0 \le x \le \frac{d}{2},\ 0 \le \theta \le \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$



we see:



needle intersects a line iff

$$X \le \frac{l}{2}\sin\Theta$$

thus

$$P\left(X \le \frac{l}{2}\sin\Theta\right) = \iint_{\{x \le \frac{l}{2}\sin\theta\}} \frac{4}{\pi d}\, dx\, d\theta$$

$$= \frac{4}{\pi d} \int_0^{\pi/2} \int_0^{l/2 \cdot \sin\theta} dx\, d\theta$$

$$= \frac{4}{\pi d} \int_0^{\pi/2} \frac{l}{2} \sin\theta \, d\theta$$

$$= \frac{2l}{\pi d} (-\cos\theta) \Big|_0^{\pi/2}$$

$$= -\frac{2l}{\pi d} (0 - 1) = \boxed{\frac{2l}{\pi d}}$$

Pick $l = d$, then

$$P(\text{intersection}) = \frac{2}{\pi}$$

$$\therefore \quad \pi = \frac{2}{P(\text{intersect})}$$

experiment:

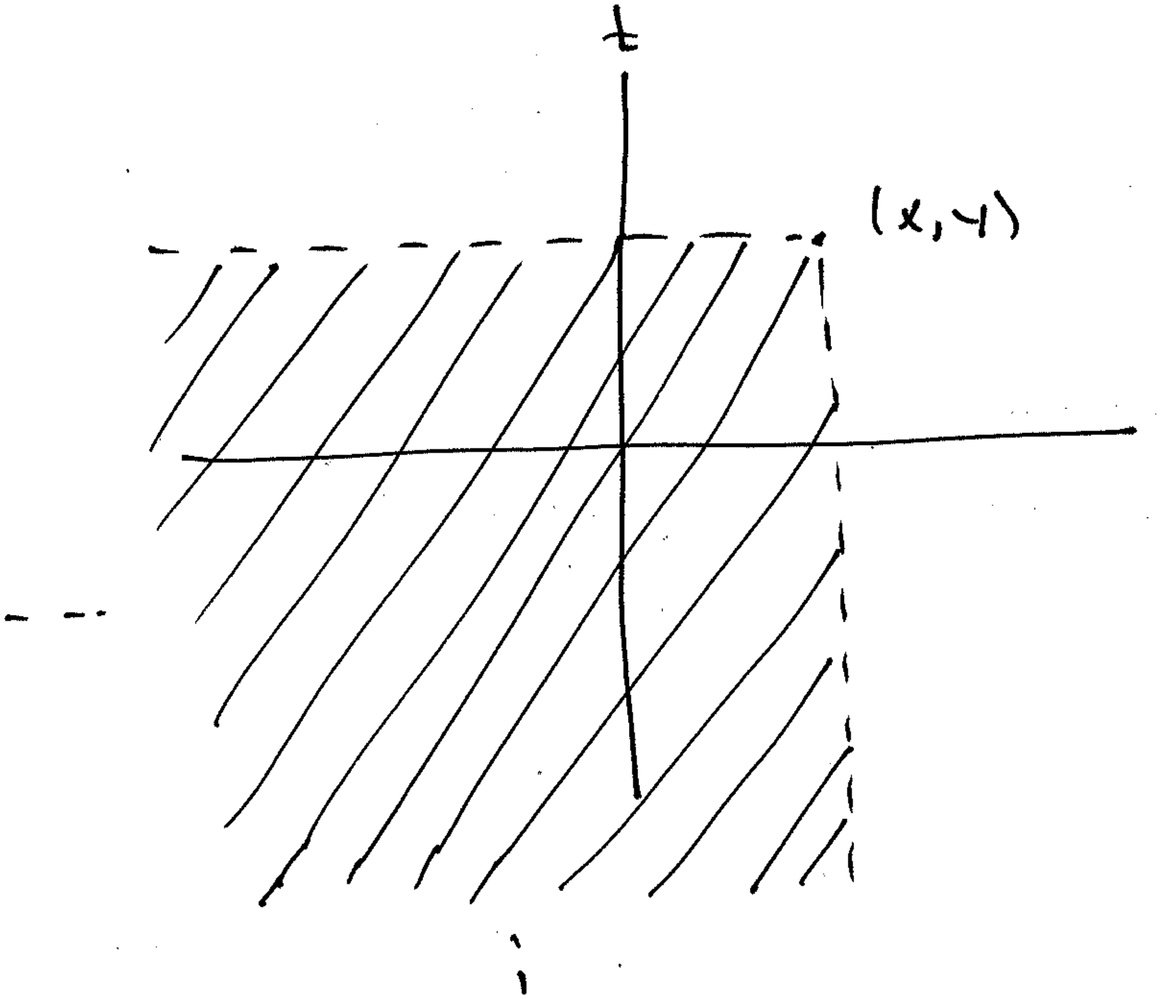
$$\pi \approx \frac{2}{\frac{\#\text{intersect}}{\#\text{needles}}} = \frac{2 \cdot \#\text{needles}}{\#\text{intersect}}.$$

Joint CDFs

If X, Y are r.v.s, their joint CDF is

$$F_{X,Y}(x,y) = P(X \le x, Y \le y)$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s,t)\, dt\, ds$$



so we can recover joint PDF by

$$f_{X,Y}(x,y) = \frac{\partial^2 F}{\partial y \partial x}(x,y).$$

Expectation

If $g: \mathbb{R}^2 \to \mathbb{R}$ we have a new r.v. $Z = g(X,Y)$. we have 2-variable expected value rule

$$E[g(X,Y)] = \iint_{\mathbb{R}^2} g(x,y) \cdot f_{X,Y}(x,y)\,dx\,dy$$

special case:

$$E[aX+bY+c] = \iint_{\mathbb{R}^2} (ax+by+c) f_{X,Y}(x,y)\,dx\,dy$$

$$= \text{-----} = aE[X] + bE[Y] + c$$

↑
exercise.

Multiple r.v.s

Joint PDF: $f_{X,Y,Z}(x,y,z)$

Joint CDF: $F_{X,Y,Z}(x,y,z)$

formulas like:

$$f_{X,Y}(x,y) = \int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z)\,dz$$

$$f_X(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z)\,dz\,dy$$

3.5 Conditioning

Suppose $A \subseteq \Omega$ and $P(A) > 0$.

Defn

the conditional PDF of X given A is a function $f_{X|A}(x)$ such that

$$P(X \in S \mid A) = \int_S f_{X|A}(x)\,dx$$

for 'any' subset $S \subseteq \mathbb{R}$.

require: $\displaystyle\int_{-\infty}^{\infty} f_{X|A}(x)\,dx = 1$