CSE 107 11-30-24

Supplemental Lecture

The sum of Poisson r.v.s

Let X, Y be independent Poisson r.v.s with Param λ_1, λ_2 resp. Then

$$Z = X + Y$$

is Poisson with Param. $\lambda_1 + \lambda_2$

Proof

we have

$$P_X(k) = e^{-\lambda_1} \cdot \frac{\lambda_1^k}{k!} \quad (k = 0, 1, 2, \ldots)$$

and

$$P_Y(k) = e^{-\lambda_2} \cdot \frac{\lambda_2^k}{k!} \quad (k = 0, 1, \ldots)$$

Thus

$$p_Z(n) = \sum_{k=0}^{\infty} p_X(k) \cdot p_Y(n-k) \quad \begin{cases} \text{require:} \\ n-k \geq 0 \\ k \leq n \end{cases}$$

$$= \sum_{k=0}^{n} e^{-\lambda_1} \cdot \frac{\lambda_1^k}{k!} \cdot e^{-\lambda_2} \cdot \frac{\lambda_2^{(n-k)}}{(n-k)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \cdot \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \cdot \lambda_1^k \cdot \lambda_2^{n-k}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \cdot \sum_{k=0}^{n} \binom{n}{k} \lambda_1^k \lambda_2^{n-k}$$

$$= e^{-(\lambda_1+\lambda_2)} \cdot \frac{(\lambda_1+\lambda_2)^n}{n!} \quad (n = 0, 1, \ldots)$$

$\therefore$ Z is Poisson with Param. $\lambda_1 + \lambda_2$.

$\blacksquare$

Ex. (6.2 #8)

During rush hour, 8-9 am, accidents occur with rate 5/hr as a Poisson Process. During period 9-11 am accidents occur as Poisson Process with rate 3/hour. What is the PMF of the total # of accidents in 8-11 am.

Soln

Let N_1 = # accidents in 8-9 am

N_2 = " " " 9-11 am

$N = N_1 + N_2$

= " " " 8-11 am

so N_1 is Poisson with $\lambda_1 = 5$, and N_2 is Poisson with $\lambda_2 = 3 \cdot 2 = 6$.

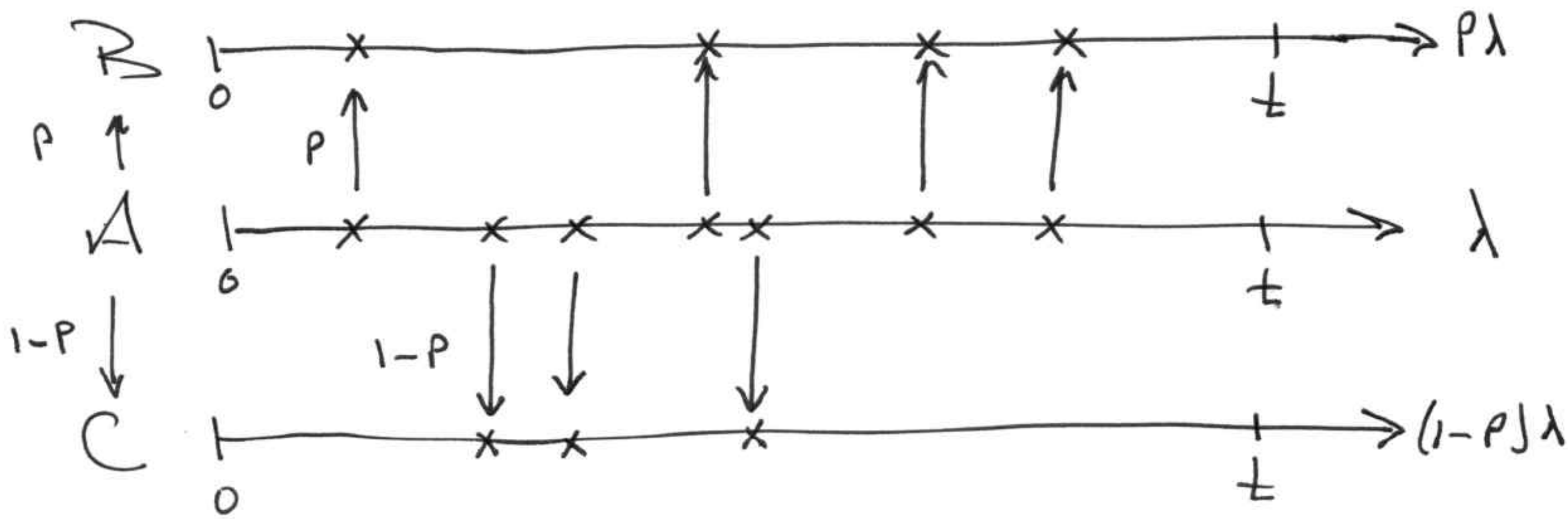
$\therefore$ N is Poisson with Param.

$$\lambda = 5 + 6 = 11$$

$$\therefore \; P_N(n) = e^{-11} \cdot \frac{11^n}{n!} \quad (n = 0, 1, 2, \ldots)$$

Splitting a Poisson Process

Let A be a Poisson Process of rate λ. Split A into two arrival Processes B, C with Probabilities p, 1−p resp.



What kind of Processes are B, C?

Let

$N = \#$ arrivals in A in $[0, t]$

$X =$ " " " B " "

$Y =$ " " " C " "

We know N is Poisson with Param. λt

$$P_N(n) = e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!} \qquad (n = 0, 1, 2, \ldots)$$

Then

$$P_X(k) = P(X = k)$$

$$= \sum_{n=k}^{\infty} P(X = k \mid N = n) \cdot P(N = n)$$

$$= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \cdot e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!}$$

$$= \sum_{n=k}^{\infty} \frac{\cancel{n!}}{k!(n-k)!} p^k (1-p)^{n-k} \cdot e^{-\lambda t} \cdot \frac{(\lambda t)^n}{\cancel{n!}}$$

$$= \frac{p^k}{k!} \cdot e^{-\lambda t} \cdot \sum_{n=k}^{\infty} \frac{(1-p)^{n-k} \cdot (\lambda t)^n}{(n-k)!} \quad \left\{ \begin{array}{l} \text{Sub} \\ n \leftarrow n+k \end{array} \right.$$

$$= \frac{p^k}{k!} e^{-\lambda t} \cdot \sum_{n=0}^{\infty} \frac{(1-p)^n (\lambda t)^{n+k}}{n!}$$

$$= \frac{(p\lambda t)^k}{k!} e^{-\lambda t} \cdot \sum_{n=0}^{\infty} \frac{((1-p)\cdot \lambda t)^n}{n!}$$

$$= \frac{(p\lambda t)^k}{k!} e^{-\lambda t} \cdot e^{(1-p)\lambda t}$$

$$= \frac{(p\lambda t)^k}{k!} \cdot e^{-\cancel{\lambda t} + \cancel{\lambda t} - p\lambda t}$$

$$= e^{-p\lambda t} \cdot \frac{(p\lambda t)^k}{k!} \qquad (k = 0, 1, 2, \ldots)$$

$\therefore$ X is a Poisson r.v. with Param. $p\lambda t$

$\therefore$ B is a Poisson Process at rate: $p\lambda$

By a similar argument C is also a Poisson Process of rate $(1-p)\lambda$.

Ex. see 6.7 #11

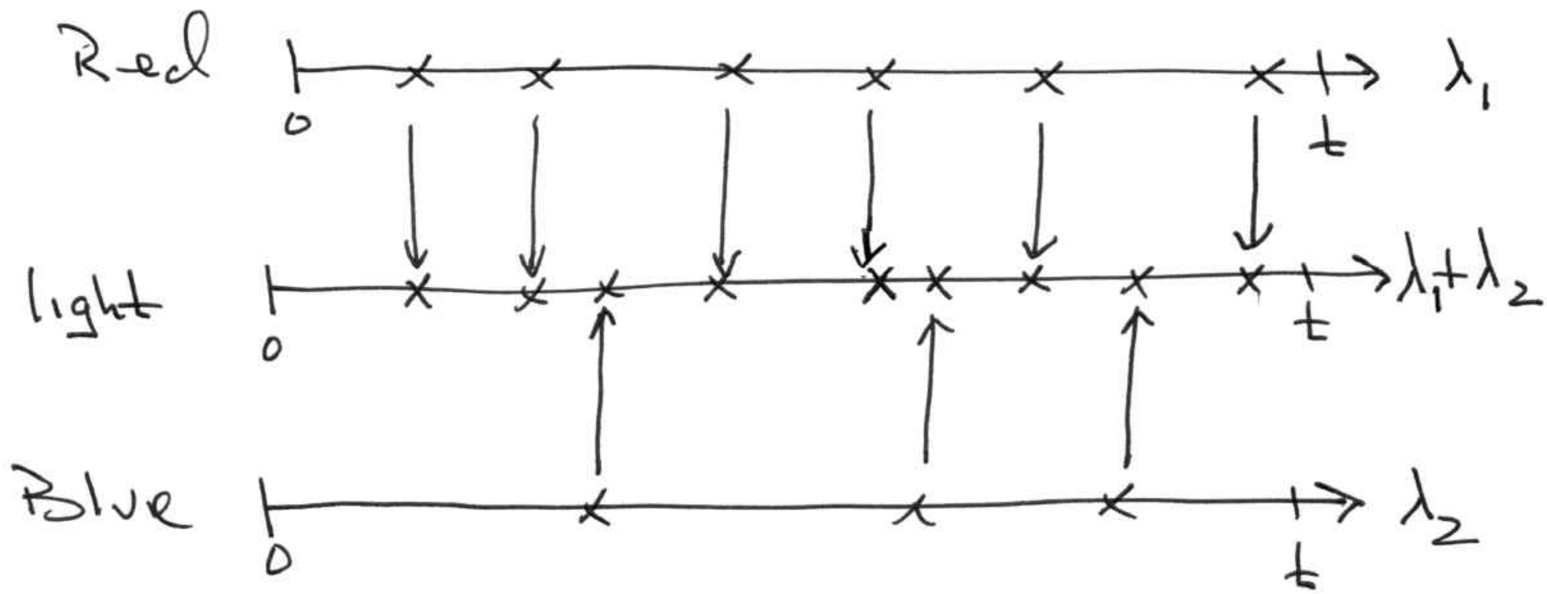
Merging Poisson Processes

Ex

An observer sees 2 lights flashing at random times, one red, one blue. The red light is a Poisson Process of rate λ_1, and blue is Poisson Processes of rate λ_2.

The observer is colorblind however, so he only sees lights.

What kind of process is the observer seeing?



Let

X = # red lights in $[0,t]$

Y = " Blue " " "

M = # lights in $[0,t]$

Know X is Poisson$(\lambda_1 t)$

Y is Poisson$(\lambda_2 t)$

Thus

$$M = X + Y$$

is Poisson with Param $\lambda_1 t + \lambda_2 t = (\lambda_1 + \lambda_2) t$

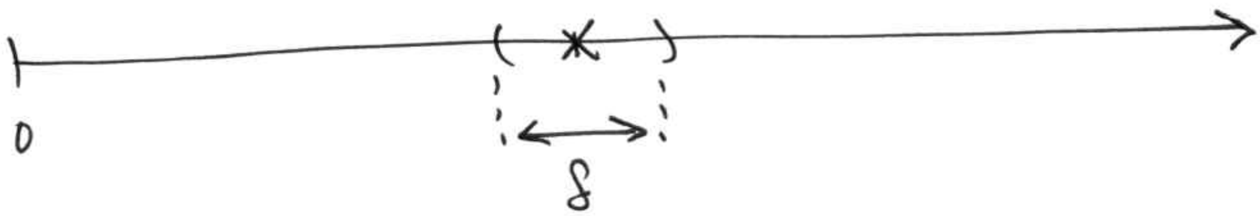
$\therefore$ M is a Poisson Process of rate $\lambda_1 + \lambda_2$.

Ex

Suppose observer sees a single flash. What is the Prob. that it was red?

solution 1

let $\delta > 0$ be the width of a small time interval containing the flash



We seek

$$P(1\text{ Red} \mid 1\text{ light})$$

$$= \frac{P(1\text{ Red}, 1\text{ light})}{P(1\text{ light})}$$

$$= \frac{P(1\text{ Red})}{P(1\text{ light})} = \frac{\cancel{\delta}\cdot\lambda_1}{\cancel{\delta}(\lambda_1+\lambda_2)} = \boxed{\frac{\lambda_1}{\lambda_1+\lambda_2}}$$

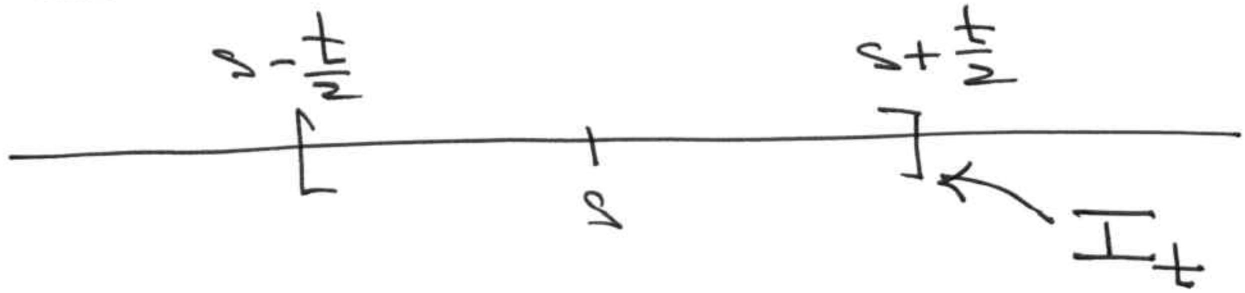
Similarly, the prob. that the colorblind observer saw a blue light is

$$P(1\text{ blue} \mid 1\text{ light}) = \boxed{\frac{\lambda_2}{\lambda_1+\lambda_2}}$$

Solution 2:

(without small interval. Probabilities)

Suppose light flashes at time s. Consider an interval of length t centered at s.



Then

$$P(1 \text{ Red in } I_t \mid 1 \text{ light in } I_t)$$

$$= \frac{P(1 \text{ Red in } I_t, 1 \text{ light in } I_t)}{P(1 \text{ light in } I_t)}$$

$$= \frac{P(1 \text{ Red in } I_t)}{P(1 \text{ light in } I_t)}$$

$$= \frac{e^{-\lambda_1 t} \cdot \frac{(\lambda_1 t)^1}{1!}}{e^{-(\lambda_1+\lambda_2)t} \cdot \frac{((\lambda_1+\lambda_2)t)^1}{1!}}$$

$$= e^{\lambda_2 t} \cdot \frac{\lambda_1}{\lambda_1+\lambda_2} \xrightarrow[\text{as } t \to 0]{} \boxed{\frac{\lambda_1}{\lambda_1+\lambda_2}}$$

$\blacksquare$