6.2 Poisson Process

Summarize Bernoulli Process

- $X_1, X_2, \ldots\ldots$ $\left\{ \begin{array}{l} \text{(i.i.d)} \\ \text{independent} \\ \text{identically} \\ \text{distributed} \\ \text{Bernoulli r.v.s} \end{array} \right.$
- Y_k = time to k^{th} success (arrival)
- $T_k = Y_k - Y_{k-1}$ = k^{th} interarrival time
- $N_t = X_1 + \ldots + X_t$ = # of arrivals in $\{1, 2, \ldots, t\}$

We know!

$$P_{Y_k}(t) = \binom{t-1}{k-1} \cdot p^k (1-p)^{t-k} \qquad t = k, k+1, \dots$$

$$P_{T_k}(t) = (1-p)^{t-1} p \qquad t = 1, 2, \dots$$

$$P_{N_t}(k) = \binom{t}{k} p^k (1-p)^{t-k} \qquad k = 0, 1, 2, \dots$$

For a continuous time arrival process we define

$$P(k, \tau) = P(\text{\# of arrivals in a time interval of len. } \tau \text{ equal to } k)$$

<u>Defn</u>

Such an arrival Process is a <u>Poisson Process</u> with rate $\lambda > 0$ iff

(a) <u>time homogeneity</u>

$P(k, \tau)$ is the same for all intervals of length τ.

(b) <u>independence</u>

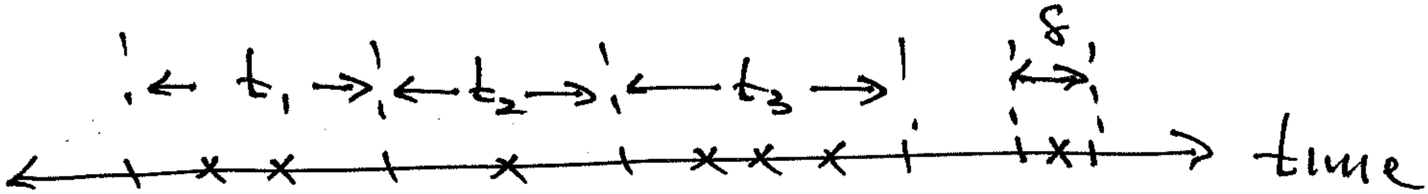
The # of arrivals in disjoint time intervals are independent.

(e) Small interval Probabilities

let $\delta > 0$ be small. Then

$$P(k,\delta) \approx \begin{cases} 1-\lambda\delta & \text{if } k=0 \\ \lambda\delta & \text{if } k=1 \\ 0 & \text{if } k=2,3,\ldots \end{cases}$$

we call λ the arrival rate



more Precisely

$$P(k,\delta) = \begin{Bmatrix} 1-\lambda\delta & k=0 \\ \lambda\delta & k=1 \\ 0 & k=2,3\ldots \end{Bmatrix} + o(\delta)$$

where $o(\delta)$ denotes a function with the property

$$\lim_{\delta \to 0} \frac{o(\delta)}{\delta} = 0$$

For instance, the function $c\delta^2$ has this property.

Observe that

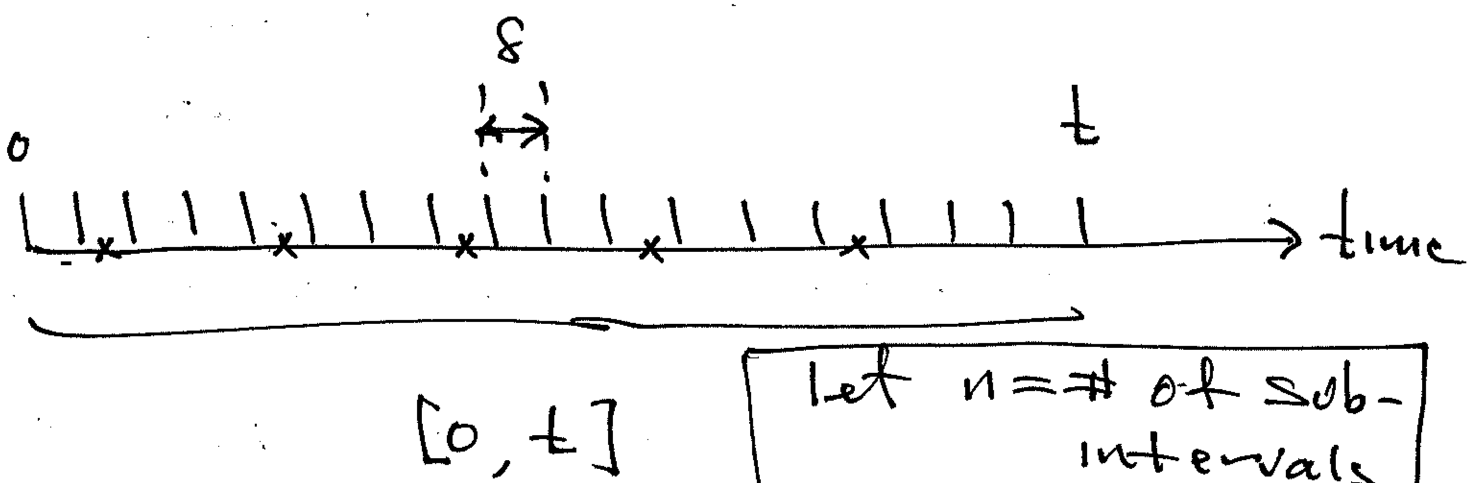
$$E[\#\text{arrivals in } [t, t+\delta]]$$

$$= (1-\lambda\delta)\cdot 0 + \lambda\delta \cdot 1$$

$$= \lambda\delta$$

$\therefore$ λ is expected # arrivals per unit time.

To reason about a large time interval, we divide into small intervals



$\boxed{\text{let } n = \#\text{ of sub-intervals so } \frac{t}{n} = \delta}$

choose δ so small that prob. of more than one arrival in one small interval is negligible

We now have (approximately) a Bernoulli Process with

$$p = \lambda \delta = \frac{\lambda t}{n}$$

note $\delta \to 0$ iff $n \to \infty$. so

P(k arrivals in $[0,t]$)

$$= \binom{n}{k} \left(\frac{\lambda t}{n}\right)^{k} \left(1 - \frac{\lambda t}{n}\right)^{n-k} \quad \Big\{ \text{Binomial dist.}$$

In limit as $n \to \infty$ ($\delta \to 0$) this becomes Poisson dist. with Param. λt. Thus

P(k arrivals in $[0,t]$)

$$= e^{-\lambda t} \cdot \frac{(\lambda t)^{k}}{k!}$$

Let N_t denote the # of arrivals of the Poisson Process in $[0, t]$, then

$$P_{N_t}(k) = e^{-\lambda t} \cdot \frac{(\lambda t)^k}{k!} \quad (k = 0, 1, \ldots)$$

We have

$$E[N_t] = \lambda t$$

$$Var(N_t) = \lambda t$$

Let T be time to 1^{st} arrival. We derive the distribution of T. First note

$$\{T > t\} = \{N_t = 0\}$$

Thus

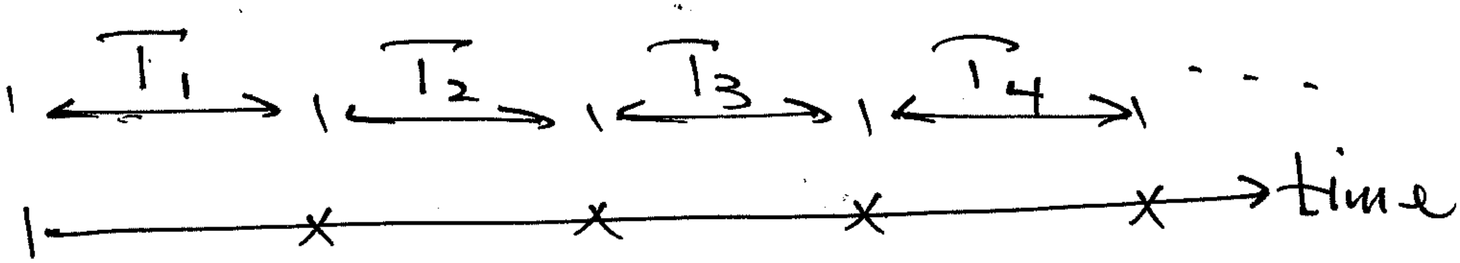
$$\begin{aligned} F_T(t) &= P(T \le t) \\ &= 1 - P(T > t) \\ &= 1 - P(N_t = 0) \\ &= 1 - P_{N_t}(0) \\ &= 1 - e^{-\lambda t} \end{aligned}$$

Now diff. w.r.t t to get

$$f_T(t) = \lambda e^{-\lambda t} \quad (t \geq 0)$$

Thus T is exponential with parameter λ.

Now let T_k be the k^{th} inter-arrival time



Recall, the exponential r.v. is memoryless

Thus each T_K is exponential with Param. λ

$$f_{T_K}(t) = \lambda e^{-\lambda t} \quad (t \geq 0)$$

Now define

$$Y_K = T_1 + T_2 + \cdots + T_K \quad (K=1,2,\ldots)$$

i.e. Y_K = time to K^{th} arrival.

The dist. of Y_K is derived in the text (P. 316-317), called <u>Erlang PDF of order K</u>

$$f_{Y_K}(t) = \frac{\lambda^K \cdot t^{K-1} \cdot e^{-\lambda t}}{(K-1)!} \qquad (t \geq 0)$$

Exercise

Prove that the sum of K independent exponential r.v.s (of param. λ) is Erlang of order K (param λ)

Hint use induction on K

Base: $Y_1 = T_1$

Induction: $Y_K = Y_{K-1} + T_K$

lets summarize analogy

	Bernoulli	Poisson
time:	$t \in \{1,2,3,\dots\}$	$t \in [0,\infty)$
rate:	p/trial	λ/unit time
N_t:	Binomial(t,p)	Poisson(λt)
T_K:	Geometric(p)	Exponential(λ)
Y_K:	Pascal(p,K)	Erlang(λ,K)

Ex.

You receive emails according to a Poisson Process at rate

$$\lambda = 2 \text{ msg./hour}$$

You currently have no messages.

a.) What is Probability that you have 3 new messages when you check 30 min from now?

$$P(N_{\frac{1}{2}} = 3) = P_{N_{\frac{1}{2}}}(3)$$

$$= e^{-2\cdot\frac{1}{2}} \cdot \frac{(2\cdot\frac{1}{2})^3}{3!} = e^{-1} \cdot \frac{1}{6} = \boxed{.0613}$$

b.) What is Prob. of no messages?

$P(N_{\frac{1}{2}}=0) = P_{N_{\frac{1}{2}}}(0)$

$= e^{-2\cdot\frac{1}{2}} \cdot \frac{(2\cdot\frac{1}{2})^0}{0!}$

$= e^{-1} = \boxed{.3679}$

c.) What is expected time to the 5th new message?

$E[Y_5] = E[T_1 + \cdots + T_5]$

$= 5 \cdot \frac{1}{\lambda} = \frac{5}{2} = \boxed{2.5 \text{ hours}}$

The sum of Poisson r.v.s

Let X, Y be ind. Poisson with param. λ_1, λ_2 resp. We will show that

$$Z = X + Y$$

is also Poisson with param. $\lambda_1 + \lambda_2$.

Proof later ...