CSE 107 11-19-24

mid 2 exam: read Ed Discussion!

6.1 The Bernoulli Process

Defn

The Bernoulli Process is a sequence

$$X_1, X_2, \ldots\ldots$$

of independent Bernoulli r.v.s, with identical parameter: p.

for each $t = 1, 2, \ldots$

$$P(X_t = 1) = p \qquad P(X_t = 0) = 1 - p$$

1 is success / arrival / heads

0 is ... / ... / ...

can think of the seq. in 2 ways

- a single experiment repeated ∞-many times: $\Omega = \{0, 1\}$

or

- one experiment with $\Omega = \{$infinite bit strings$\}$

In the 2^{nd} case, assume $0 < p < 1$, and let

$$\omega = 1111\ldots\ldots \quad \text{(all 1's)}$$

we have

$$P(\omega) = P(\forall t : X_t = 1)$$

$$\leq P(X_t = 1 \text{ for } 1 \leq t \leq n)$$

$$= p^n \xrightarrow[\text{as } n\to\infty]{} 0$$

$\therefore P(\omega) = 0$.

In fact, for any $\omega \in \Omega$, we have $P(\omega) = 0$.

Memorylessness

Let T the time (# of trials) until 1^{st} success. Then T is geometric with Param. p

$$P_T(t) = (1-p)^{t-1} \cdot p \qquad (t = 1, 2, \ldots)$$

and

$$E[T] = \frac{1}{p}, \quad Var(T) = \frac{1-p}{p^2}$$

Suppose we observe

$$X_1, X_2, \ldots\ldots, X_n$$

are all failures. Then the event $\{T > n\}$ has occurred.

What is the distribution of $T-n$, the time left (after n trials) until 1^{st} head.

$$P(T-n=t \mid T>n) = \frac{P(T-n=t,\ T>n)}{P(T>n)}$$

$$= \frac{P(T=n+t,\ T>n)}{P(T>n)}$$

$$= \frac{P(T=n+t)}{P(T>n)} \quad \text{since } \{T=n+t\} \subseteq \{T>n\}$$

$$= \frac{(1-p)^{n+t-1}\cdot p}{(1-p)^n} = \frac{\cancel{(1-p)^n}\cdot(1-p)^{t-1}\cdot p}{\cancel{(1-p)^n}}$$

$$= (1-p)^{t-1}\cdot p = P(T=t).$$

so independence implies memorylessness, i.e. "fresh start"

Inter-arrival times

Let

T_1 = time to 1^{st} arrival

T_2 = time from T_1 to 2^{nd} arrival

$\vdots$

T_k = time from T_{k-1} to k^{th} arrival

$\vdots$

Also let

$$Y_k = T_1 + T_2 + \dots + T_k$$

= time (from beginning) to k^{th} arrival.

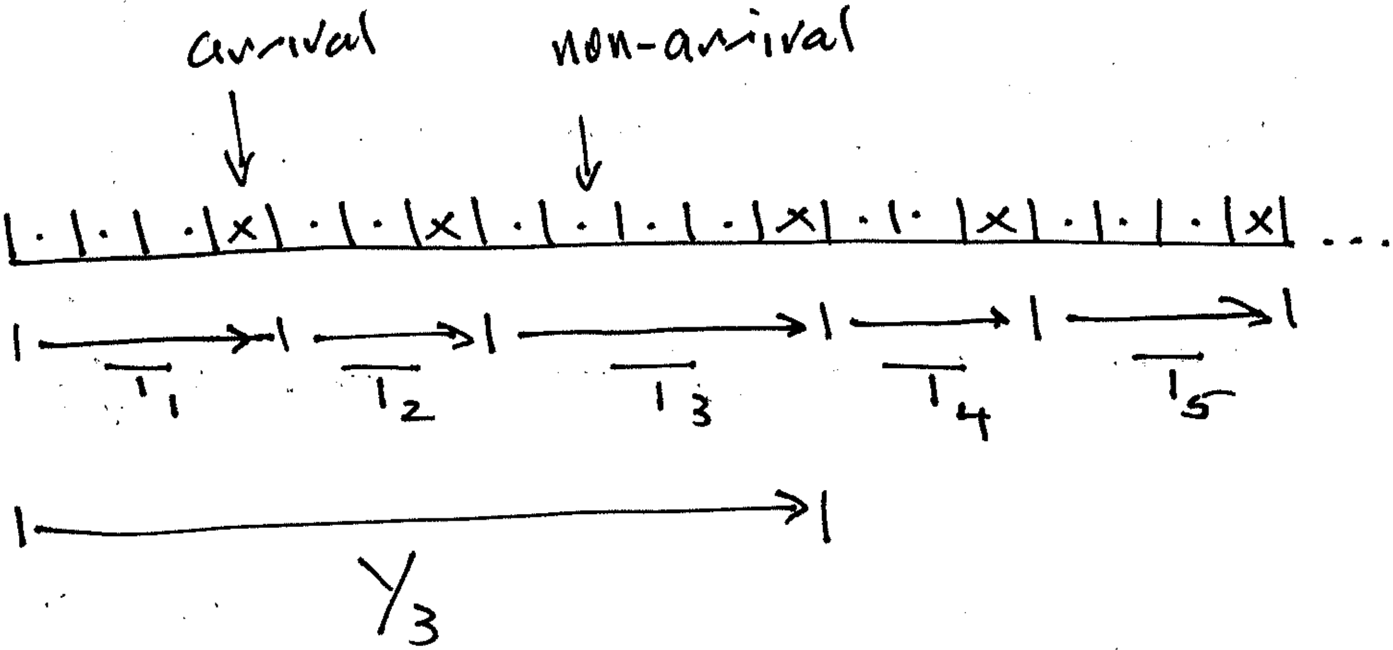
Alternately, can define

$$T_1 = Y_1$$

$$T_2 = Y_2 - Y_1$$

$$\vdots$$

$$T_k = Y_k - Y_{k-1}$$



The T_k are independent geometric with param. p.

What is the dist. of Y_K?

<u>One way</u>:

compute PMF of Y_K using

$$Y_K = Y_{K-1} + T_K$$

and convolutions (and induction)

<u>Exercise</u> do this

<u>Instead</u>: use independence

$$P_{Y_K}(t) = P(Y_K = t)$$

$$= P(K-1 \text{ arrivals in } [1, t-1], \text{ 1 arrival at time } t)$$

$$= \underbrace{P(k-1 \text{ arrivals in } [1, t-1])}_{\substack{\text{Binomical with Param:} \\ n = t-1,\ p \text{ evaluated} \\ \text{at } k-1}} \cdot \underbrace{P\begin{pmatrix}1 \text{ arrival} \\ \text{at time } t\end{pmatrix}}_{p}$$

$$= \binom{t-1}{k-1} \cdot p^{k-1} \cdot (1-p)^{t-k} \cdot p$$

Thus

$$P_{Y_k}(t) = \begin{cases} \binom{t-1}{k-1} p^k (1-p)^{t-k} & \text{if } t = k, k+1, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Called: Pascal distribution of of order k.

also

$$E[Y_k] = E[T_1 + \cdots + T_k]$$

$$= E[T_1] + \cdots + E[T_k]$$

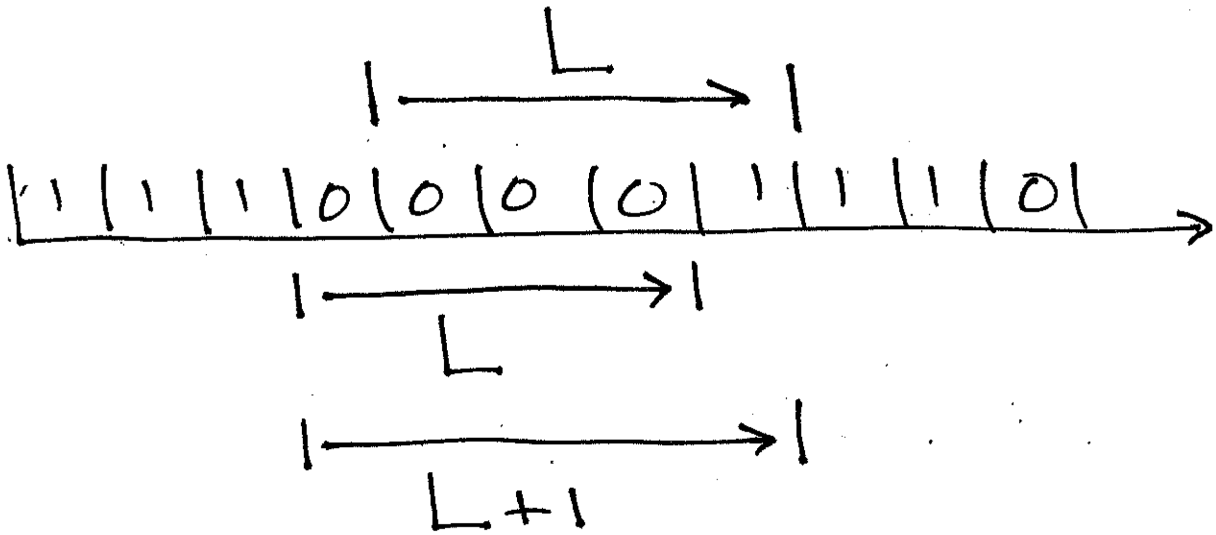
$$= \frac{k}{p}$$

$$Var(Y_k) = Var(T_1) + \cdots + Var(T_k)$$

$$= \frac{k(1-p)}{p^2}$$

<u>EX</u>

You buy a lottery ticket every day. let L be the length of your first losing streak. what is distribution of L.



one might think that $L+1$ is geometric, hence L is shifted geometric. But L cannot be 0, so $L+1$ cannot be 1, and geometric can be 1.

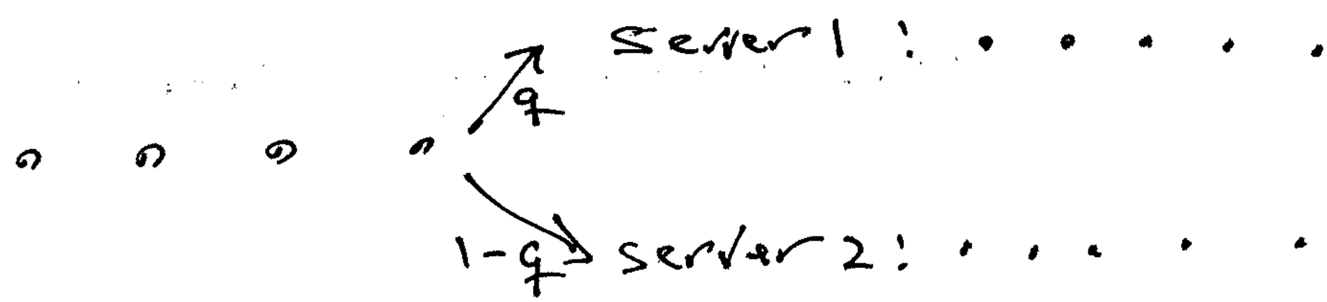
However L <u>is</u> geometric.

Splitting a Bernoulli Process

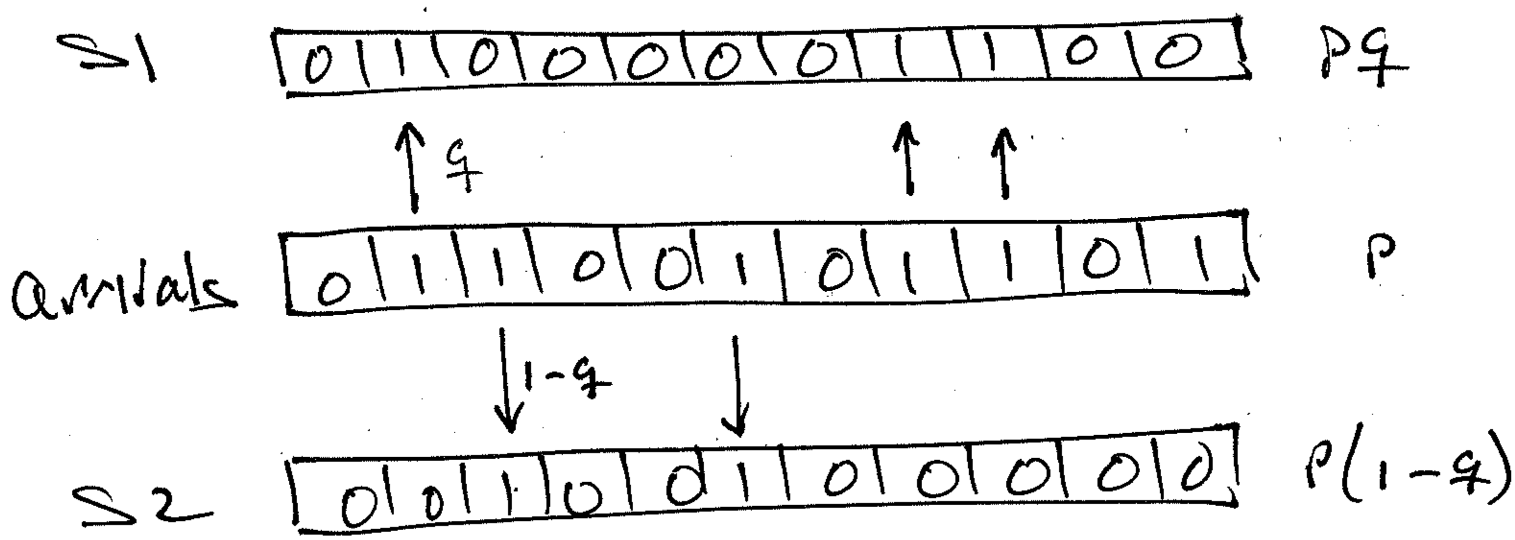
Suppose have a Bernoulli Proc. of Param. p, giving a stream of Jobs into a queue.

Send each Job to one of two servers with

$$\left.\begin{aligned} P(\text{server } 1) &= q \\ P(\text{server } 2) &= 1-q \end{aligned}\right\} \text{ independent of everything}$$



What kind of Process is received by each server ?



Then

$P(\text{arrival in } S1) = pq$

$P(\text{arrival in } S2) = p(1-q)$

Since all decisions (job/no job, S1/S2) are independent, S1 and S2 are Bernoulli with param. pq, $p(1-q)$ resp.

Merging 2 Bernoulli Processes

we have 2 independent, simultaneous Bernoulli Processes.

		Parameter
S1	: $X_1, X_2, X_3, \ldots$	p
S2	: $Y_1, Y_2, Y_3, \ldots$	q

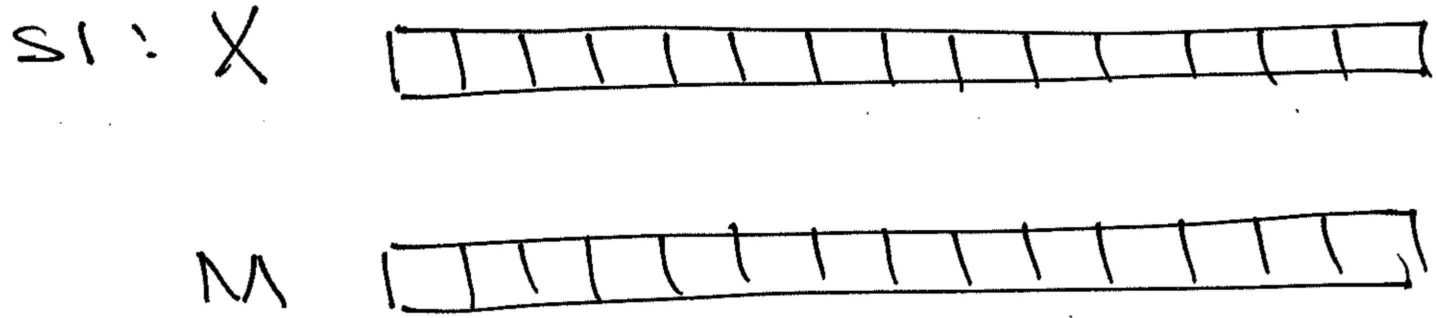
we merge S1 and S2 into a new Process M

$$M : M_1, M_2, M_3 \ldots$$

according to the following rule

$$M_t = \begin{cases} 1 & \text{if either } X_t = 1 \text{ or } Y_t = 1 \\ 0 & \text{otherwise} \end{cases}$$

What kind of process is M?



⋮

<u>later</u>