CSE 107 11-15-24

Supplemental Lecture

Ex.

Toss a coin n times, where $P(H)=p$.

let

$$X = \#\text{heads}$$

and

$$Y = \#\text{tails}$$

compute $\rho(X,Y)$.

Solution

observe $X+Y=n$, so

$$E[X]+E[Y]=E[X+Y]=E[n]=n$$

also

$$\mathrm{Var}(Y) = \mathrm{Var}(-X+n)$$

$$= (-1)^2\,\mathrm{Var}(X) = \mathrm{Var}(X)$$

also

$$(X-E[X]) + (Y-E[Y])$$

$$= (X+Y) - (E[X+Y]) = n-n = 0$$

so

$$(Y-E[Y]) = -(X-E[X])$$

Thus

$$\mathrm{Cov}(X,Y) = E[(X-E[X])(Y-E[Y])]$$

$$= -E[(X-E[X])^2]$$

$$= -\mathrm{Var}(X)$$

and

$$\rho(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sqrt{Var(X)\cdot Var(Y)}}$$

$$= \frac{-Var(X)}{\sqrt{Var(X)^2}}$$

$$= \frac{-Var(X)}{Var(X)} = \boxed{-1}$$

Perfectly negatively correlated.

Variance of a sum

Let X, Y be r.v.s (not necessarily independent.) Then

$$\boxed{\begin{aligned} &Var(X+Y) \\ &= Var(X) + Var(Y) + 2\,Cov(X,Y) \end{aligned}}$$

more generally:

$$Var\left(\sum_{i=1}^{n} X_i\right)$$

$$= \sum_{i=1}^{n} Var(X_i) + \sum_{i \neq j} Cov(X_i, X_j)$$

Proof of case $n=2$

Let $U = X - E[X]$ $\therefore E[U] = 0$

and $V = Y - E[Y]$ $\therefore E[V] = 0$.

Then

$$Var(X+Y) = Var(U+V)$$

$$= E\left[\left((U+V)-0\right)^2\right]$$

$$= E\left[U^2 + V^2 + 2UV\right]$$

$$= E[U^2] + E[V^2] + 2E[UV]$$

$$= Var(X) + Var(Y) + 2\,Cov(X,Y)$$

SEE EX. #4.15 P. 221

4.3 Conditional Expectation & Variance

Recall

$$g(y) = E[X \mid Y=y]$$

is a function of y. If we substitute Y for y we get a new r.v.

$$g(Y) = E[X \mid Y] \quad \Big\{ \underline{\text{Conditional}} \quad \underline{\text{expectation}}$$

We showed

$$E\big[E[X \mid Y]\big] = E[g(Y)]$$

$$= \int_{-\infty}^{\infty} g(y) f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} E[X \mid Y=y] \cdot f_Y(y)\, dy = E[X]$$

total expectation thm

law of iterated expectations:

$$E[X] = E[E[X|Y]]$$

We can do something similar with variance. define

$$h(y) = Var(X \mid Y=y)$$

$$= E\left[(X - E[X|Y=y])^2 \mid Y=y\right]$$

By substituting Y for y we get a new r.v.

$$Var(X|Y) = h(Y)$$

$$= E[(X-E[X|Y])^2 \mid Y]$$

called the conditional variance.

Law of total variance

$$\boxed{Var(X) = E[Var(X|Y)] + Var(E[X|Y])}$$

see proof on p. 225-227 of text.

Proof

We have

$$Var(X|Y) = E[X^2|Y] - E[X|Y]^2$$

$$\therefore E[X^2|Y] = Var(X|Y) + E[X|Y]^2$$

By iterated expectation, we have

$$Var(X) = E[X^2] - E[X]^2$$

$$= E[E[X^2|Y]] - E[E[X|Y]]^2$$

$$= E\left[\mathrm{Var}(X|Y) + E[X|Y]^2\right] - E\left[E[X|Y]\right]^2$$

$$= E\left[\mathrm{Var}(X|Y)\right] + E\left[E[X|Y]^2\right] - E\left[E[X|Y]\right]^2$$

$$= E\left[\mathrm{Var}(X|Y)\right] + \mathrm{Var}\left(E[X|Y]\right)$$

$\blacksquare$

SKIP 4.4 : Transforms

4.5 Sum of a random # of r.v.s

Let $X_1, X_2, \ldots$ be an (infinite) sequence of independent and identically distributed (i.i.d) r.v.s, i.e. all have same PDF or PMF denoted

$$f_X(x) \quad \text{or} \quad p_X(x)$$

Let N be a r.v. with $\text{range}(N) = \{0, 1, 2, \ldots\}$. Assume $\{N, X_1, X_2, \ldots\}$ is independent. Define

$$Y = X_1 + X_2 + \cdots + X_N$$

denote by $E[X]$, and $Var(X)$ the common mean & variance of all X_i $(i=1,2,3,\dots)$

Let $n \geq 0$ be fixed. Then

$$E[Y|N=n]$$
$$= E[X_1+\cdots+X_N | N=n]$$
$$= E[X_1+\cdots+X_n | N=n]$$
$$= E[X_1+\cdots+X_n] \quad \{ \text{ by indep.}$$
$$= n E[X]$$

Therefore

$$E[Y|N] = N E[X]$$

by law of iterated expectations

$$E[Y] = E[E[Y|N]]$$

$$= E[N \cdot E[X]]$$

$$= E[N] \cdot E[X]$$

Also

$$\mathrm{Var}(Y|N=n)$$

$$= \mathrm{Var}(X_1 + \cdots + X_N | N=n)$$

$$= \mathrm{Var}(X_1 + \cdots + X_n | N=n)$$

$$= \mathrm{Var}(X_1 + \cdots + X_n) \quad \text{by indep.}$$

$$= \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n)$$

$$= n \cdot \mathrm{Var}(X).$$

replace n by N to get

$$Var(Y|N) = N \cdot Var(X)$$

By law of total variance

$$Var(Y) = E[Var(Y|N)] + Var(E[Y|N])$$

$$= E[N\, Var(X)] + Var(N \cdot E[X])$$

$$= E[N] \cdot Var(X) + E[X]^2 \cdot Var(N)$$

Ex

suppose we have 2 coins

coin 1 : $P(H) = p$

coin 2 : $P(H) = q$

we flip coin 1 until 1st head, let N be # flips. now flip coin 2 N times, and let Y be # heads in coin 2 sequence. observe

$$Y = X_1 + \cdots + X_N$$

each X_i is Bernoulli with parameter q. All coins are indep. let $E[X]$, $Var(X)$ denote common mean & variance of X_i. So

$$E[N] = \frac{1}{p}, \quad Var(N) = \frac{1-p}{p^2}$$

geometric

and

$$E[X] = q, \quad Var(X) = q(1-q)$$

Therefore

$$E[Y] = E[N] \cdot E[X] = \boxed{\frac{q}{p}}$$

and

$$Var(Y) = E[N]\, Var(X) + E[X]^2 \cdot Var(N)$$

$$= \boxed{\frac{q(1-q)}{p} + \frac{q^2(1-p)}{p^2}}$$