CSE 107 11-12-24

continue last ex. . . .

d.) find $E[Y]$.

<u>Solu</u>

$$E[Y] = \int_{-\infty}^{\infty} y \cdot f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} y \cdot \frac{1}{2} \int_{1}^{3} \frac{e^{-(y-x)^2/2}}{\sqrt{2\pi}}\, dx\, dy$$

$$= \frac{1}{2} \int_{1}^{3} \left(\int_{-\infty}^{\infty} y \frac{e^{-(y-x)^2/2}}{\sqrt{2\pi}}\, dy \right) dx$$

$$= \frac{1}{2} \int_{1}^{3} x\, dx = \frac{1}{2} \cdot \frac{1}{2} x^2 \Big|_{1}^{3} = \boxed{2}.$$

4.1 Derived Distributions

If $Y = g(X)$ how can we compute PDF of Y : $f_Y(y)$ from the PDF of X : $f_X(x)$

The 2-step Process:

① find the CDF $F_Y(y)$

$$F_Y(y) = P(g(X) \leq y)$$

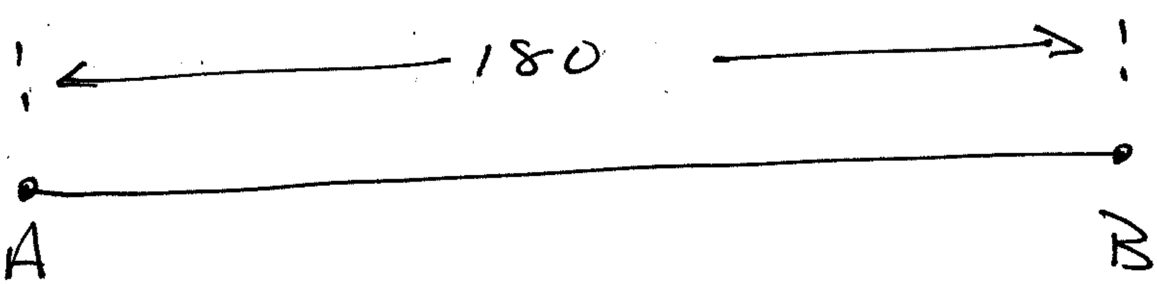
$$= \int_{\{x \mid g(x) \leq y\}} f_X(x)\,dx$$

② Differentiate

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

Ex

A car travels distance 180 mi. at a const. rate R, where R is a r.v. uniform on $[30, 60]$.



Let T be the time of travel. find $f_T(t)$.

Soln

from $180 = R \cdot T$, we get

$$T = \frac{180}{R}$$

(1) $F_T(t) = P(T \le t)$

$$= P\left(\frac{180}{R} \le t\right)$$

$$= P\left(\frac{180}{t} \le R\right)$$

$$= P\left(R \ge \frac{180}{t}\right)$$

$$= 1 - P\left(R < \frac{180}{t}\right)$$

$$= 1 - F_R\left(\frac{180}{t}\right)$$

② diff. w.r.t. t :

$$f_T(t) = -f_R\left(\frac{180}{t}\right)\cdot\left(-\frac{180}{t^2}\right)$$

We are given

$$f_R(r) = \begin{cases} \frac{1}{30} & 30 \le r \le 60 \\ 0 & \text{otherwise} \end{cases}$$

so

$$f_T(t) = \begin{cases} -\frac{1}{30}\cdot\left(-\frac{180}{t^2}\right) & \text{if } 30 \le \frac{180}{t} \le 60 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{6}{t^2} & \text{if } \frac{1}{60} \le \frac{t}{180} \le \frac{1}{30} \\ 0 & \text{otherwise} \end{cases}$$

hence

$$f_T(t) = \begin{cases} \frac{6}{t^2} & \text{if } 3 \le t \le 6 \\ 0 & \text{otherwise} \end{cases}$$

If $Y = aX + b$, we have

$$\boxed{f_Y(y) = \frac{1}{|a|} \cdot f_X\left(\frac{y-b}{a}\right) \qquad (a \neq 0)}$$

Proof

① $F_Y(y) = P(Y \le y)$

$= P(aX + b \le y)$

$= \begin{cases} P\left(X \le \frac{y-b}{a}\right) & \text{if } a > 0 \\ P\left(X \ge \frac{y-b}{a}\right) & \text{if } a < 0 \end{cases}$

$$= \begin{cases} P\left(X \le \frac{y-b}{a}\right) & \text{if } a>0 \\ 1-P\left(X < \frac{y-b}{a}\right) & \text{if } a<0 \end{cases}$$

$$= \begin{cases} F_X\left(\frac{y-b}{a}\right) & a>0 \\ 1-F_X\left(\frac{y-b}{a}\right) & a<0 \end{cases}$$

Thus by differentiating:

(2)

$$f_Y(y) = \begin{cases} \frac{1}{a} f_X\left(\frac{y-b}{a}\right) & a>0 \\ -\frac{1}{a} f_X\left(\frac{y-b}{a}\right) & a<0 \end{cases}$$

$$\therefore \quad f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right).$$

■

Ex.

Let X be normal, mean $= \mu$ and variance $= \sigma^2$. Then

$$Y = aX + b$$

is also normal with mean $= a\mu + b$ and variance $= a^2\sigma^2$

Proof

We know

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-(x-\mu)^2/2\sigma^2}$$

Thus

$$f_Y(y) = \frac{1}{|a|} \cdot f_X\left(\frac{y-b}{a}\right)$$

$$= \frac{1}{\sqrt{2\pi} \cdot |a| \sigma} \cdot e^{-\left(\frac{y-b}{a} - \mu\right)^2 / 2\sigma^2}$$

$$= \frac{1}{\sqrt{2\pi} |a| \sigma} \cdot e^{-(y-b-a\mu) / 2a^2\sigma^2}$$

$$= \frac{1}{\sqrt{2\pi} \cdot |a| \sigma} \cdot e^{-(y-(a\mu+b))^2 / 2(a\sigma)^2}$$

$\blacksquare$

<u>Defn</u>

$g : \mathbb{R} \to \mathbb{R}$ is <u>strictly monotonic</u> iff

(1) $x_1 < x_2 \implies g(x_1) < g(x_2)$ (increasing)

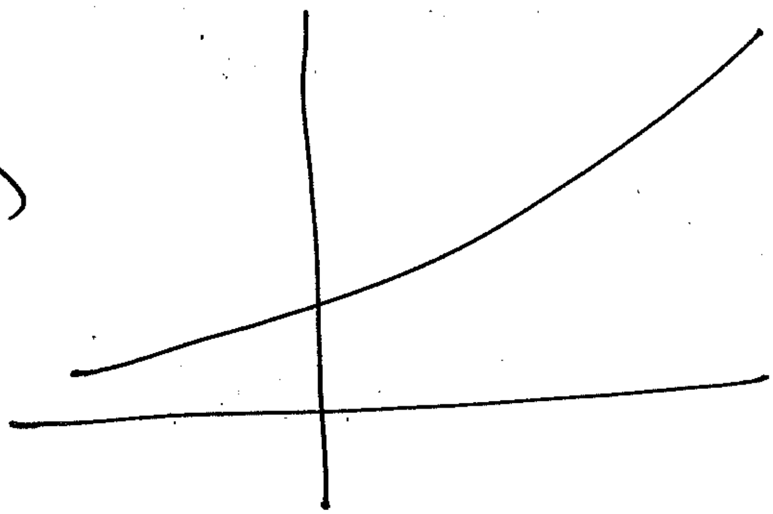
or

(2) $x_1 < x_2 \implies g(x_1) > g(x_2)$ (decreasing)

for all $x_1, x_2 \in \mathbb{R}$.

such functions are bijective

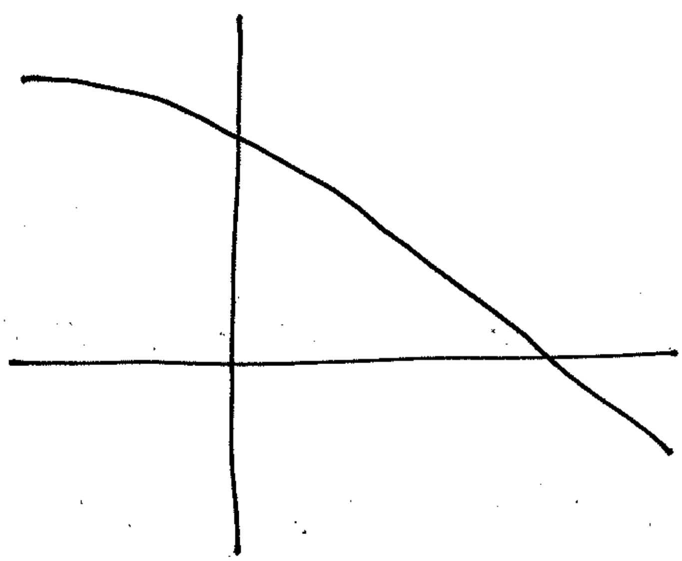
(1)



g^{-1} exists

increasing

(2)



g^{-1} exists.

decreasing

Let $h = g^{-1}$, i.e.

$$h(g(x)) = x \quad \text{for} \quad x \in \mathbb{R}$$

and

$$g(h(y)) = y \quad \text{for} \quad y \in \mathbb{R}$$

exercise

$$(1) \Rightarrow h = g^{-1} \text{ is increasing}$$

and

$$(2) \Rightarrow h = g^{-1} \text{ is decreasing}$$

If g is differentiable, so is h, and

$$(1) \Leftrightarrow g' > 0 \Leftrightarrow h' > 0$$

and

$$(2) \Leftrightarrow g' < 0 \Leftrightarrow h' < 0$$

Theorem

Let $Y = g(X)$, where g is strictly monotonic. Then

$$f_Y(y) = f_X(h(y)) \cdot |h'(y)| .$$

Proof

Suppose g is strictly increasing, so h is increasing. Then

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(g(X) \le y) \\ &= P(X \le h(y)) \\ &= F_X(h(y)) \end{aligned}$$

$$\therefore \quad f_Y(y) = f_X(h(y)) \cdot h'(y)$$

$$= f_X(h(y)) \cdot |h'(y)| \quad \text{since } h' > 0.$$

<u>exercise</u>: do the case where g is decreasing. $\blacksquare$

<u>Ex.</u>

Suppose $Y = X^2$ where X is a cont. r.v. with $\text{range}(X) \subseteq [0, \infty)$.

Then also

$$\text{range}(Y) \subseteq [0, \infty)$$

note $g(x) = x^2$ is strictly increasing on $[0,\infty)$, and so is its inverse $h(y) = \sqrt{y}$. Thus

$$f_Y(y) = f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} \quad (y \geq 0)$$

Two random variables: $Z = g(X,Y)$

Ex.

let X, Y be independent, uniform r.v. on $[0,1]$. Thus

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

and likewise for Y. Define a r.v. Z by

$$Z = \max(X, Y)$$

Then for $0 \le z \le 1$ we have

$$F_Z(z) = P(Z \le z)$$

$$= P(\max(X, Y) \le z)$$

$$= P(X \le z, Y \le z)$$

$$= P(X \le z) \cdot P(Y \le z) \quad \Big\{ \text{by ind.}$$

$$= F_X(z) \cdot F_Y(z)$$

$$= z^2$$

$$\therefore f_Z(z) = \begin{cases} 2z & 0 \le z \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Exercise

Let X be cont. unif. on $[a,b]$

and Y cont. unif. on $[c,d]$.

Find PDF of

$$Z = \max(X, Y)$$

assume X, Y are independent

note: if $b<c$, then $Z=Y$, or

if $d<a$, then $Z=X$. so

assume both $b \geq c$ and $d \geq a$,

so $[a,b] \cap [c,d] \neq \phi$.

Sums and convolutions

<u>Goal</u>: find PDF (or PMF)

of

$$Z = X + Y$$

in terms of PDFs (or PMFs)
of X and Y.