CSE 107 11-1-24

Supplemental Lecture

A normal r.v. Y is called <u>standard normal</u> iff its mean is $\mu=0$, and its variance is $\sigma^2=1$.

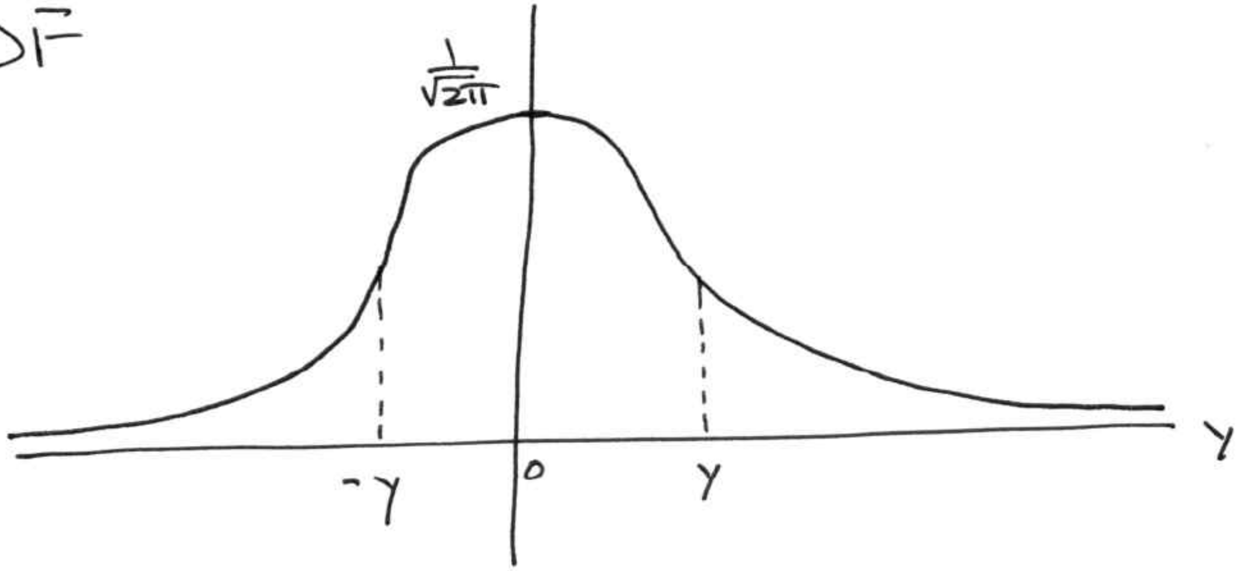
$$\boxed{f_Y(y) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}}}$$

The CDF of the std. normal is denoted

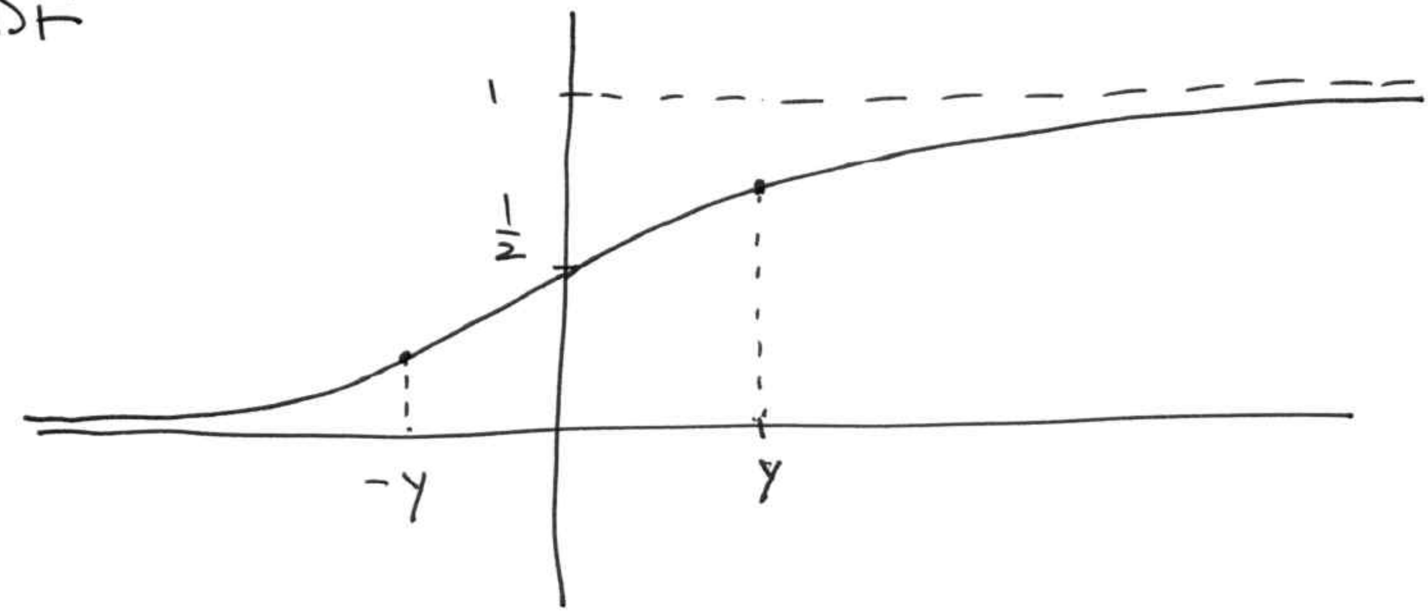
$$\Phi(y) = P(Y \le y)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{t^2}{2}}\,dt$$

PDF



CDF



The <u>standard normal table</u> gives values of $\Phi(y)$ for

$$0 \le y \le 3.49$$

If $y > 3.49$, then $\Phi(y) \approx 1$

If $y < 0$, the identity

$$\Phi(-y) = 1 - \Phi(y) \qquad (y \in \mathbb{R})$$

allows us to compute $\Phi(y)$

Ex.

- $\Phi(1.84) = .9671$

- $\Phi(-1.84) = 1 - \Phi(1.84)$

$$= 1 - (.9671) = .0329$$

If X is normal with mean & variance μ, σ^2 we can compute $F_X(x)$ as follows

Let

$$Y = \frac{X-\mu}{\sigma} = \frac{1}{\sigma}\cdot X - \frac{\mu}{\sigma}$$

Then Y is normal with

$$E[Y] = \frac{1}{\sigma}\mu - \frac{\mu}{\sigma} = 0 \quad \checkmark$$

and

$$Var(Y) = \frac{1}{\sigma^2} Var(X) = \frac{1}{\sigma^2}\cdot\sigma^2 = 1 \quad \checkmark$$

so Y is standard normal. Thus

$$\begin{aligned}
F_X(x) &= P(X \le x) \\
&= P\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right) \\
&= P\left(Y \le \frac{x-\mu}{\sigma}\right) \\
&= \Phi\left(\frac{x-\mu}{\sigma}\right)
\end{aligned}$$

Ex.

The mean high temp. in Santa Cruz on Nov. 1 is 66° with std. dev. 4.4°. What is the probability that the high temp. will be $\leq 57^\circ$

Solution

Let X be the high temp. Then

$$F_X(57) = P(X \leq 57)$$

$$= P\left(\frac{X-66}{4.4} \leq \frac{57-66}{4.4}\right)$$

$$= P(Y \leq -2.04)$$

$$= \Phi(-2.04)$$

$$= 1 - \Phi(2.04)$$

$$= 1 - .9793 = \boxed{.0207}$$

Ex.

what is the Probability that a normal r.v. X is within 1 standard deviation of its mean?

Solution

$$P(\mu-\sigma \le X \le \mu+\sigma)$$

$$= P\left(\frac{(\mu-\sigma)-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{(\mu+\sigma)-\mu}{\sigma}\right)$$

$$= P(-1 \le Y \le 1)$$

$$= P(Y \le 1) - P(Y < -1)$$

$$= \Phi(1) - \Phi(-1) = \Phi(1) - (1-\Phi(1))$$

$$= 2\Phi(1) - 1 = 2(.8413) - 1$$

$$= \boxed{.6826}$$

A similar calculation shows the probability of being within 2σ of the mean is

$$2\Phi(2)-1 = 2(.9772)-1 = \boxed{.9544}$$

and for 3σ we have

$$2\Phi(3)-1 = 2(.9987)-1 = \boxed{.9974}$$