CSE 107 10-8-24

1.6 Combinatorics

read 1.6 (review of cse 16)

- # Permutations of n objects

$$n! = n\cdot(n-1)\cdot(n-2)\cdots 3\cdot 2\cdot 1$$

- # of k-Permutations of n objects

$$\frac{n!}{(n-k)!} = n(n-1)(n-2)\cdots(n-k+1)$$

- # k-combinations of n objects (without repetition)

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$= \left[\begin{array}{l}\text{\# k-subsets}\\ \text{of an n-set}\end{array}\right]$$

$$= \left[\begin{array}{l}\text{\# bit strings of len. n}\\ \text{that contain k 1's}\end{array}\right]$$

- # Partitions of n objects into k sets, with set i having n_i elements ($1 \le i \le k$, $n_1 + n_2 + \cdots + n_k = n$)

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!\, n_2! \cdots n_k!}$$

$$= \left[\begin{array}{l}\text{\# anagrams of n objects with}\\ n_i \text{ objects of type } i \ (1 \le i \le k)\end{array}\right]$$

- k-combinations of n objects, (repetition is allowed)

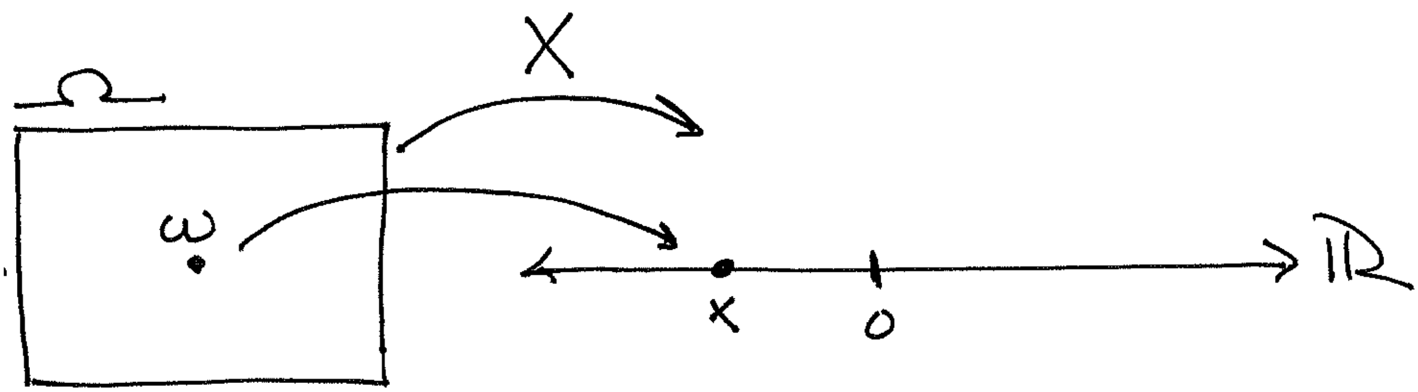
$$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$$

2.1 Random Variables

Defn

A <u>random variable</u> is a function

$$X : \Omega \to \mathbb{R}$$



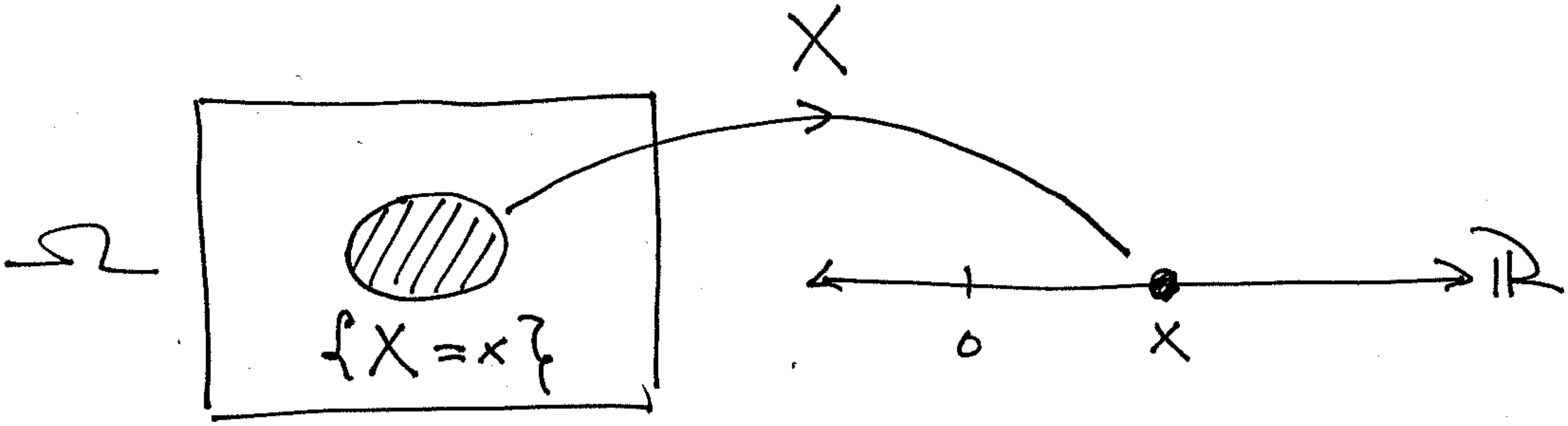
notation

always use capital letters for r.v.s, and lower case for their values:

$$X(\omega) = x$$

- we denote by $\{X=x\}$ the event

$$\{X=x\}=\{\omega\in\Omega \mid X(\omega)=x\}\subseteq\Omega$$



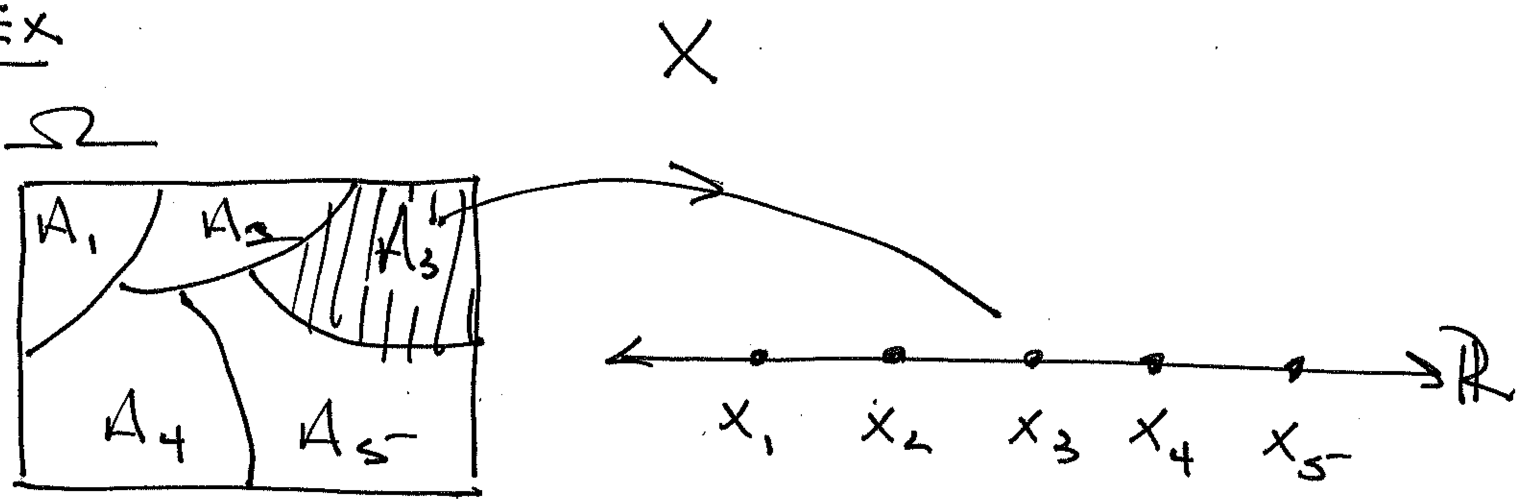
Think of X as transporting 'mass' from Ω to $\mathbb{R}$.

- a r.v. X is called <u>discrete</u> iff

$$\mathrm{range}(X)=\{X(\omega)\mid\omega\in\Omega\}\subseteq\mathbb{R}$$

is a discrete subset of $\mathbb{R}$.

Ex



- X is called continuous .... later.

Ex. Roll 2 fair independent dice.

Let

$$X((i,j)) = i + j$$

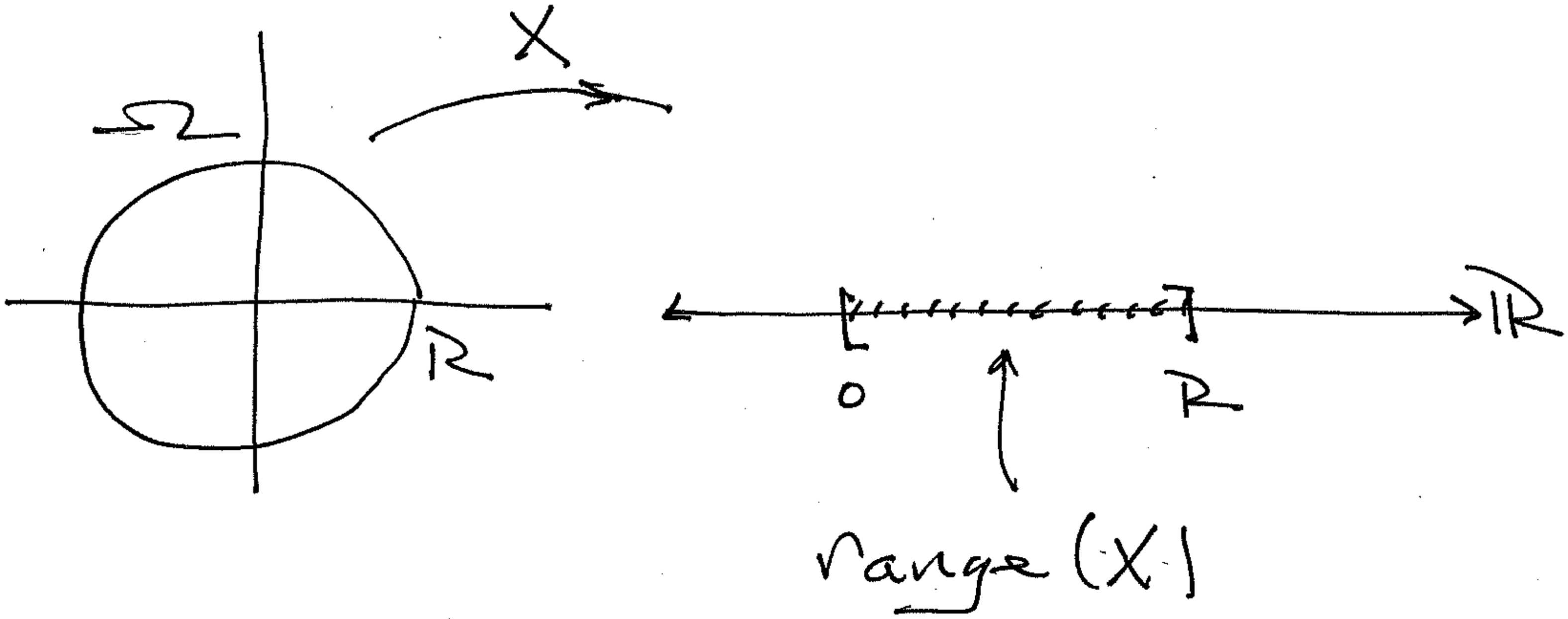
$$\Omega = \left\{ \begin{array}{cccc} 11 & 12 & 13 & \cdots 16 \\ 21 & 22 & 23 & \cdots 26 \\ \vdots & & & \vdots \\ 61 & 62 & 63 & \cdots 66 \end{array} \right\}$$

range (X)
$= \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

$$\Omega \xrightarrow[X]{} \mathbb{R}$$

$\therefore$ X is discrete

<u>Ex</u> Throw a dart at a circular target of radius R, centered at $(0,0)$. let $X((x,y)) = \sqrt{x^2+y^2}$, the distance from impact point (x,y) to center



This is a continuous r.v.

2.2 Probability mass functions

Defn

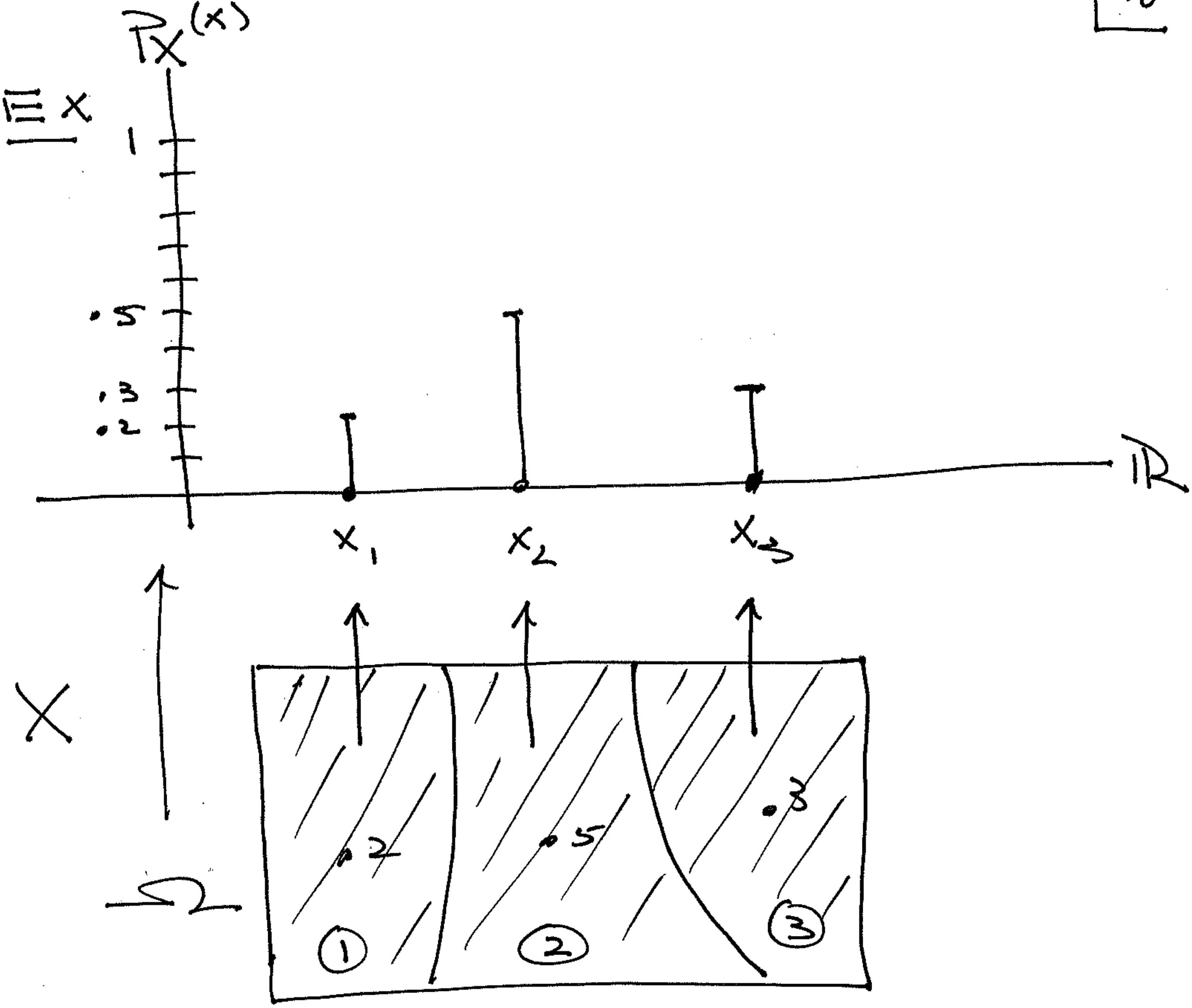
Let X be a discrete r.v. The probability mass function (PMF) of X is

$$p_X(x) = P(\{X=x\}) = P(X=x)$$

observe $p_X : \mathbb{R} \to [0,1]$

note: $\sum_{x \in \mathrm{range}(X)} p_X(x) = 1$

Ex



$$P_X(x) = \begin{cases} .2 & \text{if } x = x_1 \\ .5 & \text{if } x = x_2 \\ .3 & \text{if } x = x_3 \\ 0 & \text{otherwise} \end{cases}$$

The Bernoulli Random Variable

Perform one Bernoulli trial

$$X(\omega) = \begin{cases} 1 & \text{if } \omega \text{ is success} \\ 0 & \text{if } \omega \text{ is failure} \end{cases}$$

It's PMF is

$$P_X(k) = \begin{cases} p & \text{if } k = 1 \\ 1-p & \text{if } k = 0 \end{cases}$$

Parameter: $p \in [0,1]$.

The Binomial Random Variable

Perform a seq. of n independent Bernoulli trials, with same parameter $p \in [0,1]$.

$X(\omega) = \#$ of successes in n trials.

The PMF of X is the Binomial Probabilities in 1.6

$$P_X(k) = P(X=k)$$

$$= \binom{n}{k} p^k (1-p)^{n-k}$$

for $k = 0, 1, 2, \dots, n$

By the Binomial theorem

$$\sum_{k=0}^{n} P_X(k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= (p + (1-p))^n$$

$$= 1^n = 1$$

Parameters: $\begin{cases} n = \# \text{ flips} \\ p = \text{Prob. of heads} \end{cases}$

The Geometric Random Variable

We perform independent Bernoulli trials (Parameter p) until the first success. Let

$$X(\omega) = \#\text{ trials until } 1^{st} \text{ success}$$

We see .....