ESE 107 10-4-24

Supplemental Lecture

1.5 Independence

$A, B \subseteq \Omega$ are independent if

$$* \qquad P(A|B) = P(A)$$

The occurance of B gives no information about A. i.e.

$$\frac{P(A \cap B)}{P(B)} = P(A)$$

$$** \qquad P(A \cap B) = P(A) \cdot P(B)$$

Defn

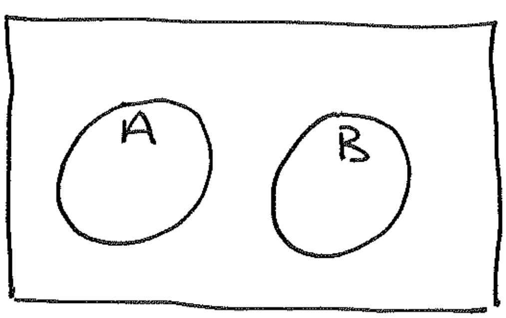
Given $A, B \subseteq \Omega$, we say A, B are independent iff

$$\boxed{P(A \cap B) = P(A) \cdot P(B)}$$

warning: $A \cap B = \phi$ does not imply independence. why?

$$A \cap B = \phi \Rightarrow P(A \cap B) = 0$$

neither $P(A)$ nor $P(B)$ need be 0 though



$A \cap B = \phi$

$P(A) \neq 0$

$P(B) \neq 0$

$P(A \cap B) = 0$

Ex. Throw two independent fair dice. let

$A_i = \{1^{st} \text{ die is } i\}$ $1 \le i \le 6$

$B_j = \{2^{nd} \text{ die is } j\}$ $1 \le j \le 6$

note: A_i, B_j are independent (any, i, j).

$$P(A_i \cap B_j) = P(A_i) \cdot P(B_j) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

This gives the discrete uniform Probability law.

Let $C = \{\text{sum} = 5\}$

$= \{14, 23, 32, 41\}$

$\therefore \; P(C) = \frac{4}{36} = \frac{1}{9}$

Is C independent of A_2 ?

$A_2 = \{21, 22, 23, 24, 25, 26\}$

$A_2 \cap C = \{23\}$ $P(A_2 \cap C) = \frac{1}{36}$

$P(A_2) \cdot P(C) = \frac{6}{36} \cdot \frac{1}{9} = \frac{1}{54} \neq \frac{1}{36} = P(A_2 \cap C)$

$\therefore$ C and A_2 are <u>not</u> independent.

Let $D = \{\text{doubles}\} = \{11, 22, 33, 44, 55, 66\}$

Is D independent of A_2 ?

$P(D) = \frac{6}{36} = \frac{1}{6}$

$D \cap A_2 = \{22\}$ $P(D \cap A_2) = \frac{1}{36}$

$P(D) \cdot P(A_2) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$

$\therefore$ D and A_2 <u>are</u> independent.

<u>Exercise</u>

Let $A, B \subseteq \Omega$. Prove that the following are equivalent.

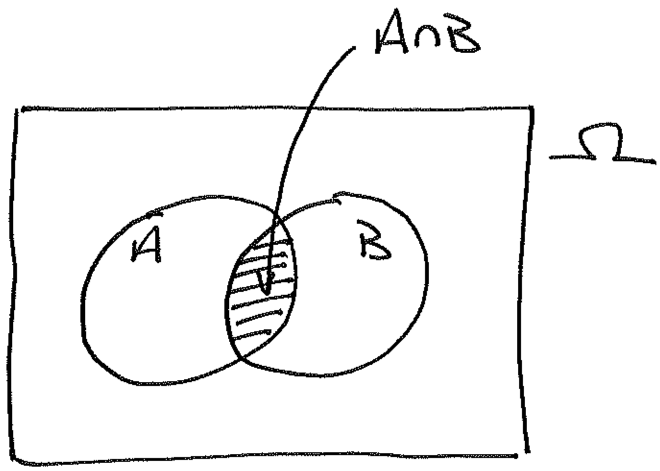
(a) A, B are independent

(b) A, B^c " "

(c) A^c, B " "

(d) A^c, B^c " "

Geometric intuition:



independence: $P(A|B) = P(A)$

i.e. $$\frac{P(A\cap B)}{P(B)} = \frac{P(A)}{P(\Omega)}$$

Conditional Independence

Recall that if $P(C) > 0$, then

$$P(\cdot \mid C)$$

is itself a valid Probability law.

Defn

we say events A, B are **conditionally independent given C** iff

$$\boxed{P(A \cap B \mid C) = P(A \mid C) \cdot P(B \mid C)}$$

note:

$$P(A \cap B \mid C) = \frac{P(A \cap B \cap C)}{P(C)}$$

$$= \frac{\cancel{P(C)} \cdot P(B \mid C) \cdot P(A \mid B \cap C)}{\cancel{P(C)}} \quad \left\{ \begin{array}{l} \text{by the} \\ \text{mult.} \\ \text{rule} \end{array} \right.$$

$$= P(B|C) \cdot P(A|B \cap C)$$

so A, B are cond. ind. given C iff

$$\cancel{P(B|C)} \cdot P(A|B \cap C) = P(A|C) \cdot \cancel{P(B|C)}$$

alternate definition:

$$P(A|B \cap C) = P(A|C)$$

Interpretation: A, B are cond. ind. given C iff: if C is known to occur, the additional knowledge that B occurs, does not change the Prob. of A.

Be warned that

$$\boxed{\begin{array}{l}\text{cond. ind.}\\ \text{of } A,B\\ \text{given } C\end{array}} \quad \begin{array}{c}\not\Longrightarrow\\[1em] \not\Longleftarrow\end{array} \quad \boxed{\begin{array}{l}\text{unconditional}\\ \text{ind. of } A,B\\ \\ \end{array}}$$

see examples 1.20, 1.21 in text (P. 37).

Independence of multiple events

Defn

We say A, B, C are <u>independent</u> iff the folling hold

$$(1)\ P(A\cap B\cap C) = P(A)\cdot P(B)\cdot P(C)$$

$$\underline{\text{Pairwise}}\ \underline{\text{independence}}\ \left\{\begin{array}{l}(2)\ P(A\cap B) = P(A)\cdot P(B)\\ (3)\ P(B\cap C) = P(B)\cdot P(C)\\ (4)\ P(A\cap C) = P(A)\cdot P(C)\end{array}\right.$$

Be warned that

$$\boxed{\begin{array}{l}\text{Pairwise} \\ \text{independence} \\ \text{of } A, B, C \\ \text{i.e. (2), (3), (4)}\end{array}} \quad \begin{array}{c}\not\Longrightarrow \\ \not\Longleftarrow\end{array} \quad \boxed{\begin{array}{l}\text{equation (1)} \\ P(A \cap B \cap C) \\ \quad = P(A) \cdot P(B) \cdot P(C)\end{array}}$$

see examples 1.22, 1.23 in text (p. 39).

Binomial Probabilities

Recall from CSE 16: Binomial Coefficients

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$= \left\{\begin{array}{l}\text{\# of } k\text{-element} \\ \text{subsets of an } n\text{-element set}\end{array}\right. \quad 0 \le k \le n$$

$$= \left\{\begin{array}{l}\text{\# of bit strings of len. } n \\ \text{having exactly } k \text{ 1's}\end{array}\right.$$

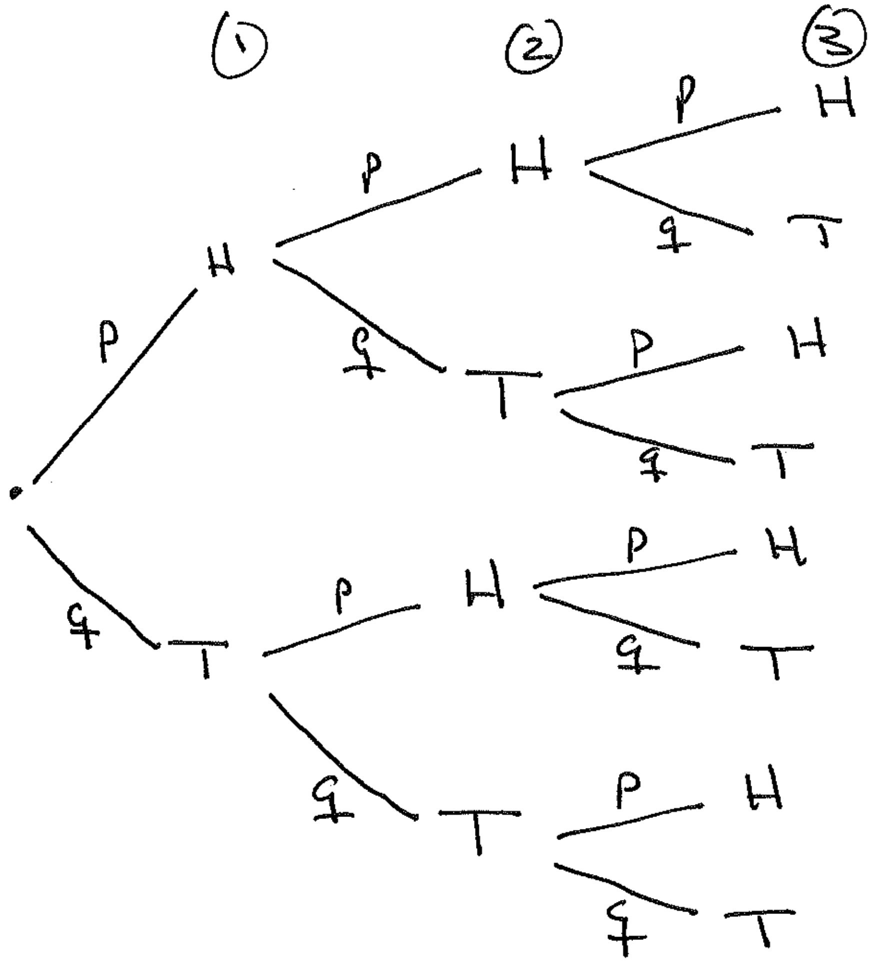
often an experiment is a composition of a seq. of identical independ sub-experiments.

A simple experiment with exactly 2 outcomes is called a __Bernoulli Trial__.

$$\text{outcomes} = \begin{cases} \text{success} = \text{heads} = 1 \\ \text{failure} = \text{tails} = 0 \end{cases}$$

__Ex__ Perform a seq. of 3 independent tosses of a weighted coin with

$$P(H) = p \quad \text{and} \quad P(T) = q = 1-p$$



outcome	Prob.
HHH	p^3
HHT	p^2q
HTH	p^2q
HTT	pq^2
THH	p^2q
THT	pq^2
TTH	pq^2
TTT	q^3

note: $p^3+3p^2q+3pq^2+q^3=(p+q)^3=1$

observe

$P(3 \text{ heads}) = p^3$

$P(2 \text{ heads}) = 3p^2q$

$P(1 \text{ head}) = 3pq^2$

$P(0 \text{ heads}) = q^3$

Generalize to n flips!

$$\boxed{P(\#\text{heads} = k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad (0 \le k \le n)}$$

Binomial Probabilities

<u>Recall</u>: Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$