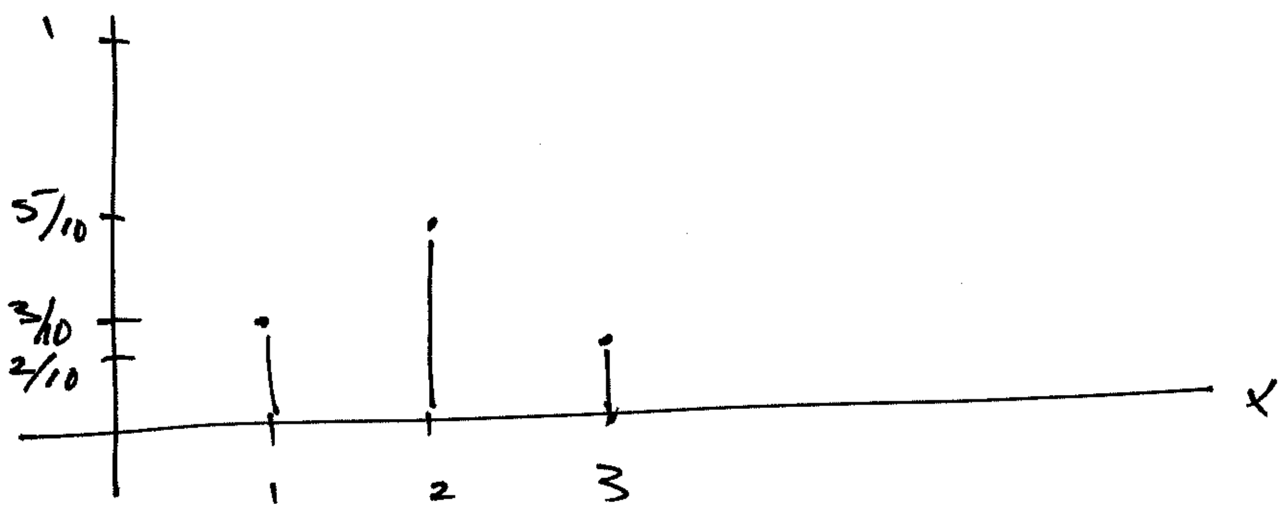
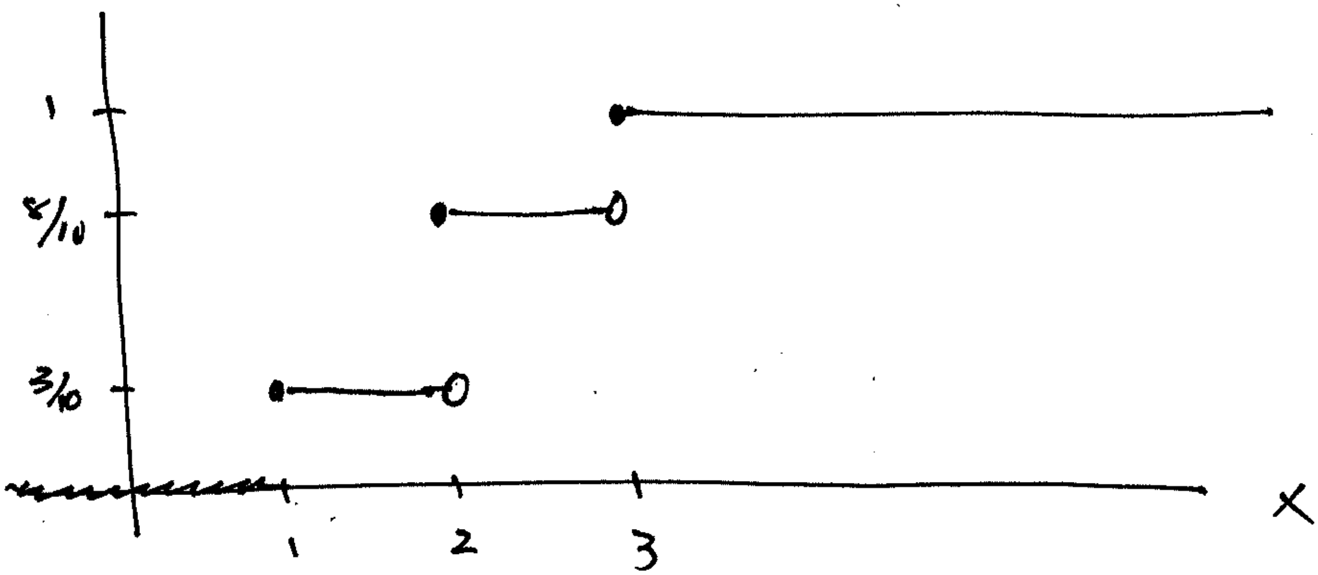
CSE 107 10-31-24

Ex. Discrete

PMF: $P_X(x) = \begin{cases} 3/10 & \text{if } k=1 \\ 5/10 & k=2 \\ 2/10 & k=3 \\ 0 & \text{otherwise} \end{cases}$

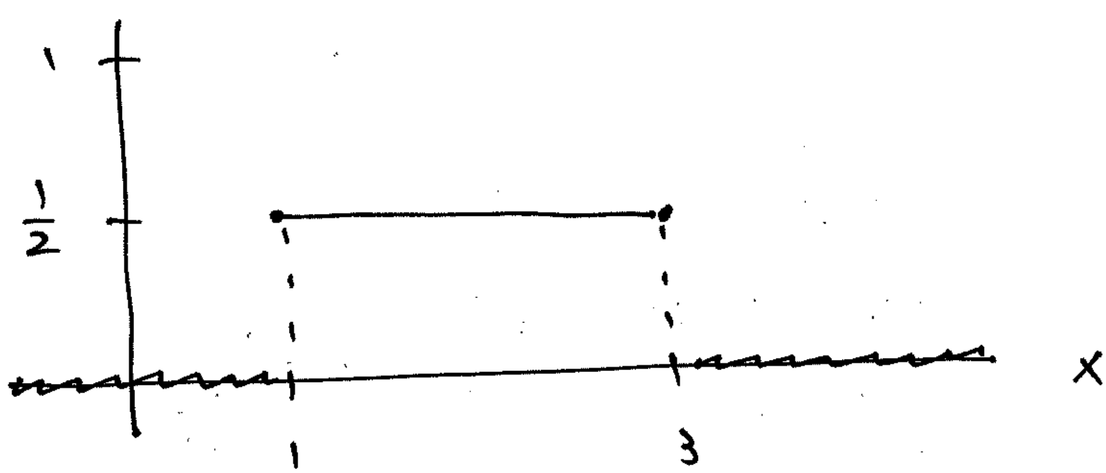


CDF: $F_X(x) = \begin{cases} 0 & \text{if } x<1 \\ 3/10 & \text{if } 1 \le x < 2 \\ 8/10 & \text{if } 2 \le x < 3 \\ 1 & \text{if } x \ge 3 \end{cases}$



<u>Ex.</u> continuous uniform on $[1, 3]$

$$\text{PDF}: \quad f_X(x) = \begin{cases} \frac{1}{2} & 1 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

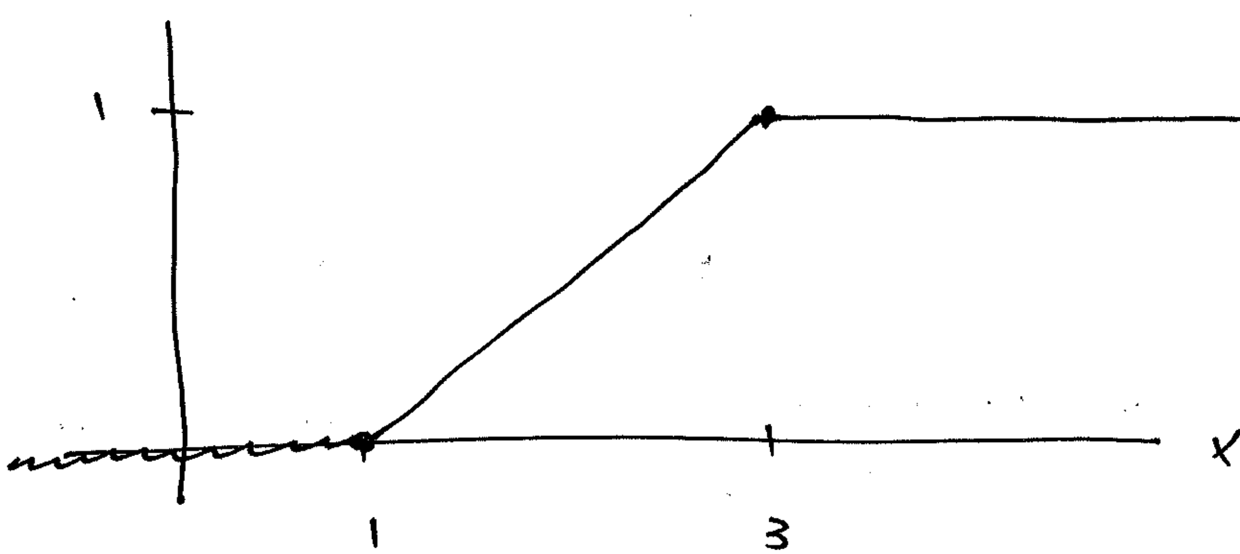


$$\text{CDF}: \quad F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

$$= \int_{1}^{x} \frac{1}{2}\,dt = \frac{1}{2}t\Big|_1^x = \frac{x-1}{2} \qquad (1 \le x \le 3)$$

so

$$\text{CDF}: \quad F_X(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{x-1}{2} & \text{if } 1 \le x < 3 \\ 1 & \text{if } x \ge 3 \end{cases}$$



Properties of CDFs

- $F_X(x) = P(X \le x)$

  is always monotonically non-decreasing.

$$x \le y \Rightarrow F_X(x) \le F_X(y)$$

- If X is discrete, $F_X(x)$ is piecewise constant

$$F_X(k) = \sum_{i \le k} P_X(i)$$

and

$$P_X(k) = P(X=k) = P(X \le k) - P(X \le k-1)$$

$$= F_X(k) - F_X(k-1)$$

(assuming range$(X) \subseteq \mathbb{Z}$).

- If X is continuous, then $F_X(x)$ is a continuous function

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

and

$$f_X(x) = \frac{d}{dx} F_X(x)$$

Ex.

You take a test 3 times, your score X is the maximum of 3 scores.

$$X = \max(X_1, X_2, X_3)$$

X_i = score on i^{th} test $i = 1, 2, 3$.

Assume scores X_1, X_2, X_3 are independent. ~~Assume~~ score X_i is uniform on $\{1,2,3,4,5,6,7,8,9,10\}$.

$$P_{X_i}(k) = \begin{cases} 1/10 & \text{if } k = 1, 2, \ldots, 10 \\ 0 & \text{otherwise.} \end{cases}$$

Find the PMF of $P_X(k)$ of X.

<u>Goal</u>: 1^{st} find $F_X(k)$, then difference to get

$$P_X(k) = F_X(k) - F_X(k-1)$$

so

$$F_X(k) = P(X \le k)$$

$$= P(X_1 \le k, X_2 \le k, X_3 \le k)$$

$$= P(X_1 \le k) \cdot P(X_2 \le k) \cdot P(X_3 \le k)$$

(by ind.)

$$= F_{X_1}(k) \cdot F_{X_2}(k) \cdot F_{X_3}(k)$$

$$= \left(\frac{k}{10}\right)\left(\frac{k}{10}\right)\left(\frac{k}{10}\right) = \left(\frac{k}{10}\right)^3$$

$$\therefore \quad P_X(k) = \begin{cases} \left(\frac{k}{10}\right)^3 - \left(\frac{k-1}{10}\right)^3 & \text{if } k = 1, 2, \ldots, 10 \\ 0 & \text{otherwise} \end{cases}$$

we compare Geometric & Exponential r.v.s using their CDFs

<u>Geometric</u> : X param. p

$$P_X(k) = p(1-p)^{k-1} \quad \text{for } k = 1, 2, 3, \ldots$$

Thus

$$F_{geo}(n) = \sum_{k=1}^{n} p(1-p)^{k-1}$$

$$= \cancel{p} \cdot \frac{1-(1-p)^n}{\cancel{1-(1-p)}}$$

$$= 1-(1-p)^n$$

Exponential: Y Param. $\lambda > 0$

$$f_Y(x) = \lambda e^{-\lambda x} \qquad (x \geq 0)$$

so for $x \geq 0$ we have

$$F_{exp}(x) = \int_0^x \lambda e^{-\lambda t}\, dt$$

$$= -e^{-\lambda t}\Big|_0^x$$

$$= -\left(e^{-\lambda x} - 1\right)$$

$$= 1 - e^{-\lambda x}$$

Pick $\delta>0$ so that

$$e^{-\lambda\delta} = 1-p \qquad (\text{i.e. } p = 1-e^{-\lambda\delta})$$

i.e. $\quad -\lambda\delta = \ln(1-p)$

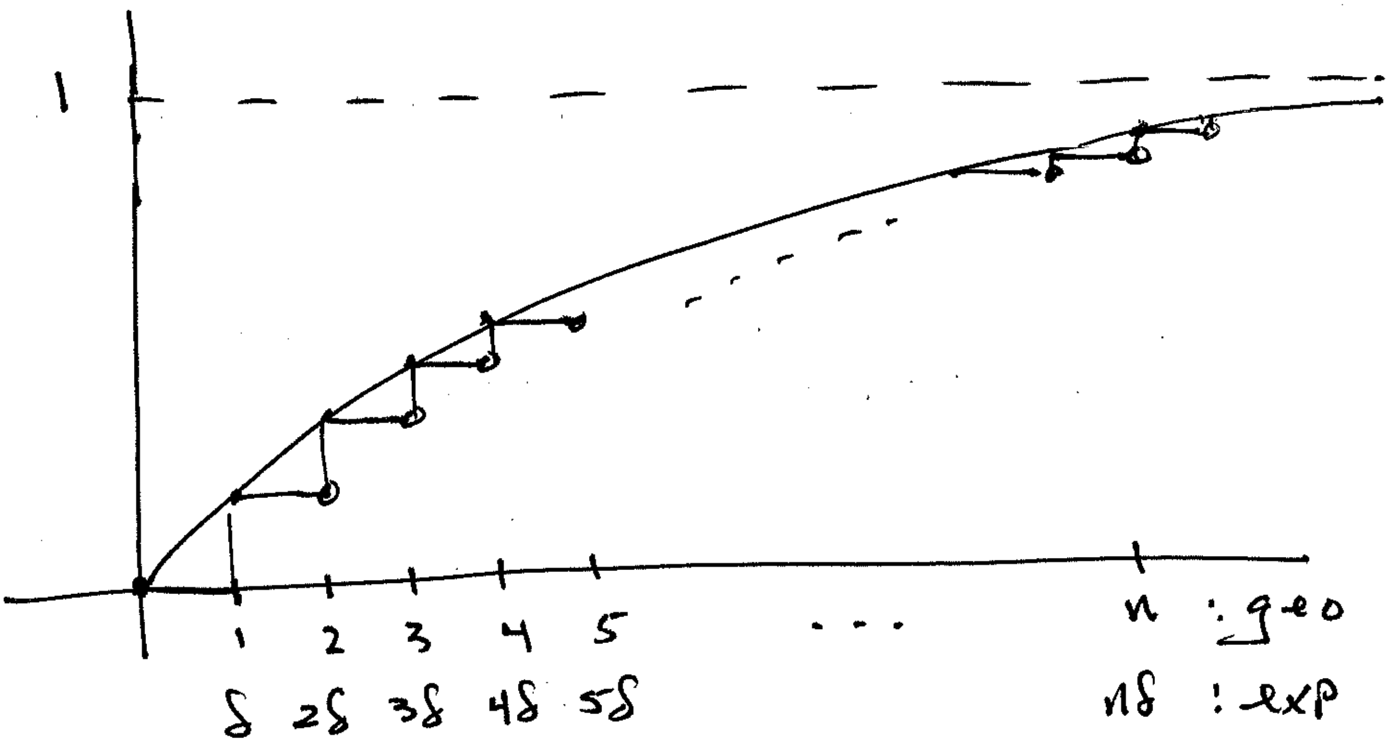
i.e. $\quad \delta = \dfrac{-\ln(1-p)}{\lambda}$

For this δ we have

$$\left(e^{-\lambda\delta}\right)^n = (1-p)^n$$

$$1-e^{-\lambda\delta n} = 1-(1-p)^n$$

$$\therefore \quad F_{exp}(\delta n) = F_{geo}(n)$$



Thus if we flip the coin fast (every δ seconds) with Prob.

$$P = 1 - e^{-\lambda\delta}$$

of heads, then the 1st time to get heads is an approx. to exponential with Param. λ.

3.3 Normal random variables

Let X be a cont. r.v. with

$$f_X(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} \cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

we call X a Normal or Gaussian rand. var.

Fact: $\displaystyle\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

see Prob. # 14 in ch.3

also see handout

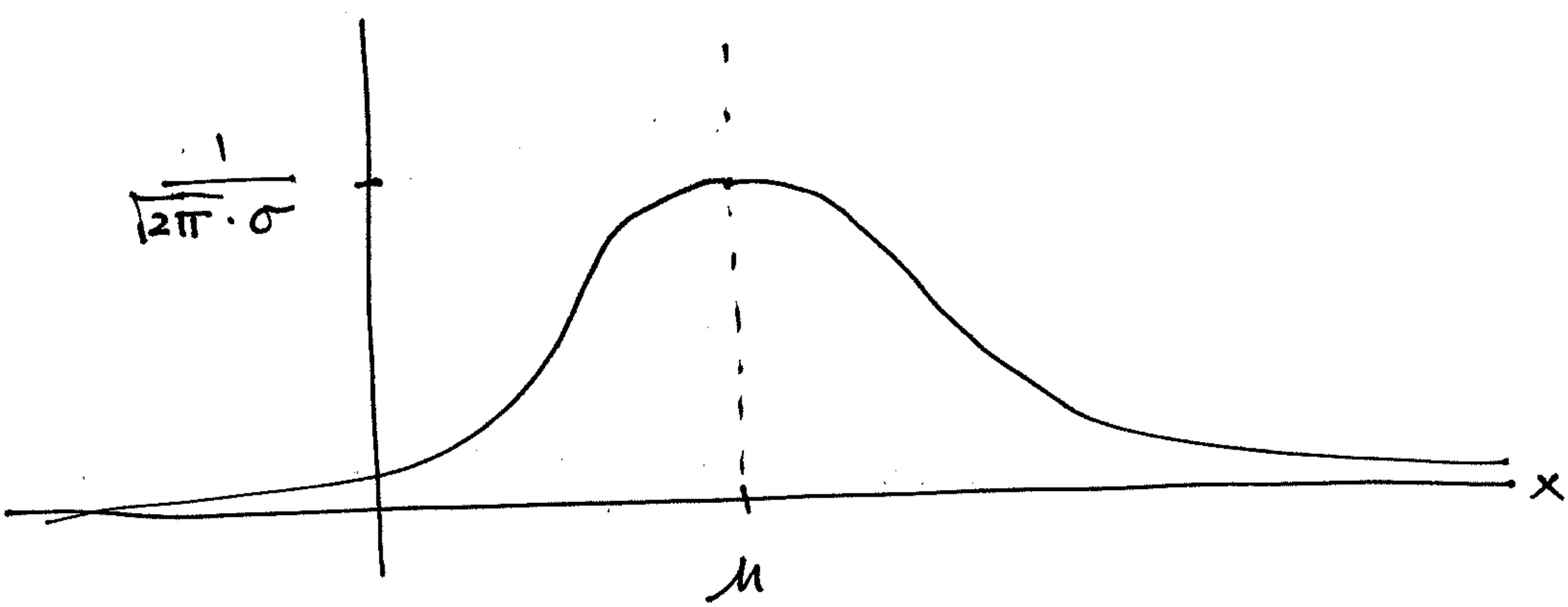
Also

$$E[X] = \mu$$

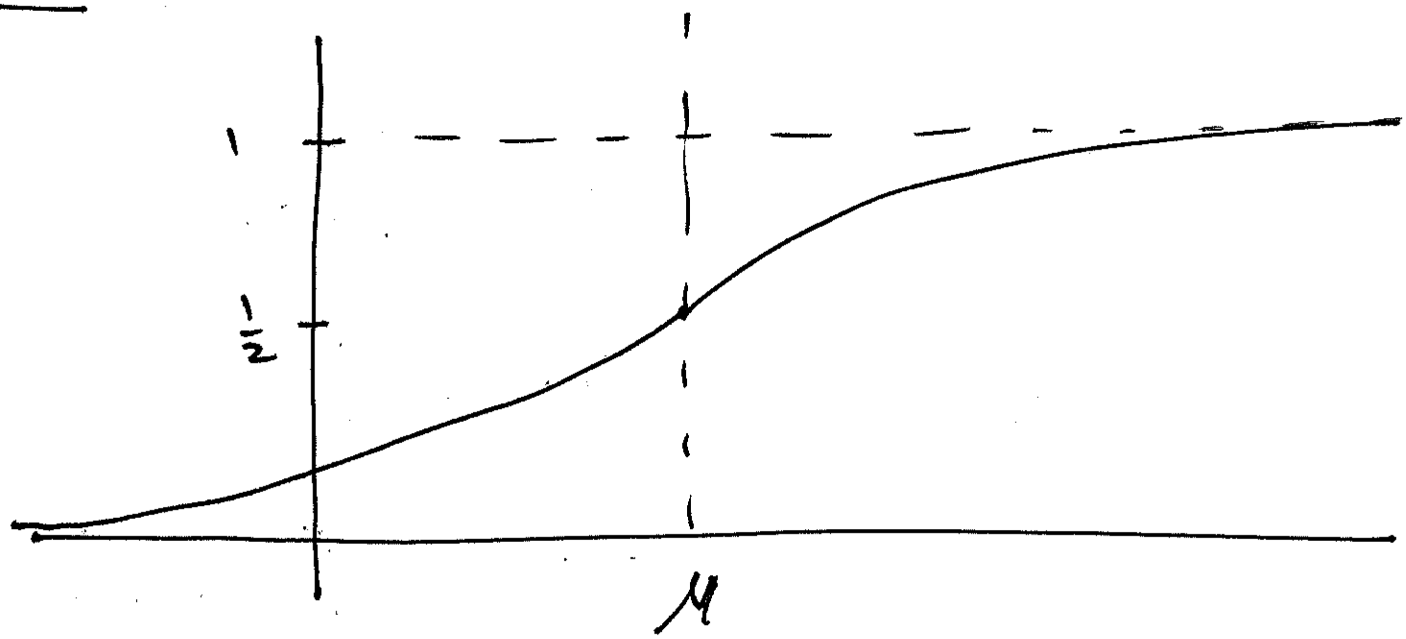
$$Var(X) = \sigma^2$$

$$\sqrt{Var(X)} = \sigma$$

PDF



CDF



It's evident that

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

is invariant under transformation

$$(x-\mu) \longleftrightarrow -(x-\mu)$$

hence is symmetric about $x=\mu$.

Theorem

If $f_X(x) = f_X(-x)$ for all $x \in \mathbb{R}$,

then $E[X] = 0$.

Proof exercise

Theorem

If X is normal with mean μ and variance σ^2, and if $a \neq 0$, then

$$Y = aX + b \text{ is } \underline{\text{also normal}}$$

with mean $a\mu + b$ and variance $a^2\sigma^2$. i.e. $E[Y] = aE[X] + b$ and

$$Var(Y) = a^2 Var(X).$$