CSE 107 10-3-24

Recal :

$$P(A \cap B) = P(A|B) \cdot P(B)$$

also:

$$P(A \cap B) = P(B|A) \cdot P(A)$$

Ex.

A random individual has Prob. .05 of having a certain disease. If a person has the disease, then Prob. of a positive test .99. If the person does -

not have disease, then

Prob. of Positive test is .10

- what is the false positive Prob.?
- what is the false negative Prob.?

Let

$T = \{\text{test is Positive}\}$

$D = \{\text{Person has disease}\}$

we are given

$P(T|D) = .99 \qquad P(T^c|D) = .01$

$P(T|D^c) = .10 \qquad P(T^c|D^c) = .90$

$P(D) = .05 \qquad P(D^c) = .95$

we must find:

$P(T \cap D^c)$ (false positive)

$P(T^c \cap D)$ (false negative)

we have

$$P(T \cap D^c) = P(T|D^c) \cdot P(D^c)$$

$$= (.10)(.95) = \boxed{.095}$$

and

$$P(T^c \cap D) = P(T^c|D) \cdot P(D)$$

$$= (.01)(.05) = \boxed{.0005}$$

<u>Recall</u>

- $P(A \cap B) = P(A) \cdot P(B|A)$

Generalize this

- $P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|A \cap B)$

<u>Proof</u>

$$RHS = P(\cancel{A}) \cdot \frac{P(\cancel{A \cap B})}{P(\cancel{A})} \cdot \frac{P(A \cap B \cap C)}{P(\cancel{A \cap B})}$$

$$= P(A \cap B \cap C) = LHS.$$

<u>Exercise</u> Prove for 4 sets

$$P(A \cap B \cap C \cap D) = P(A) \cdot P(B|A) \cdot P(C|A \cap B) \cdot P(D|A \cap B \cap C)$$

Multiplication rule

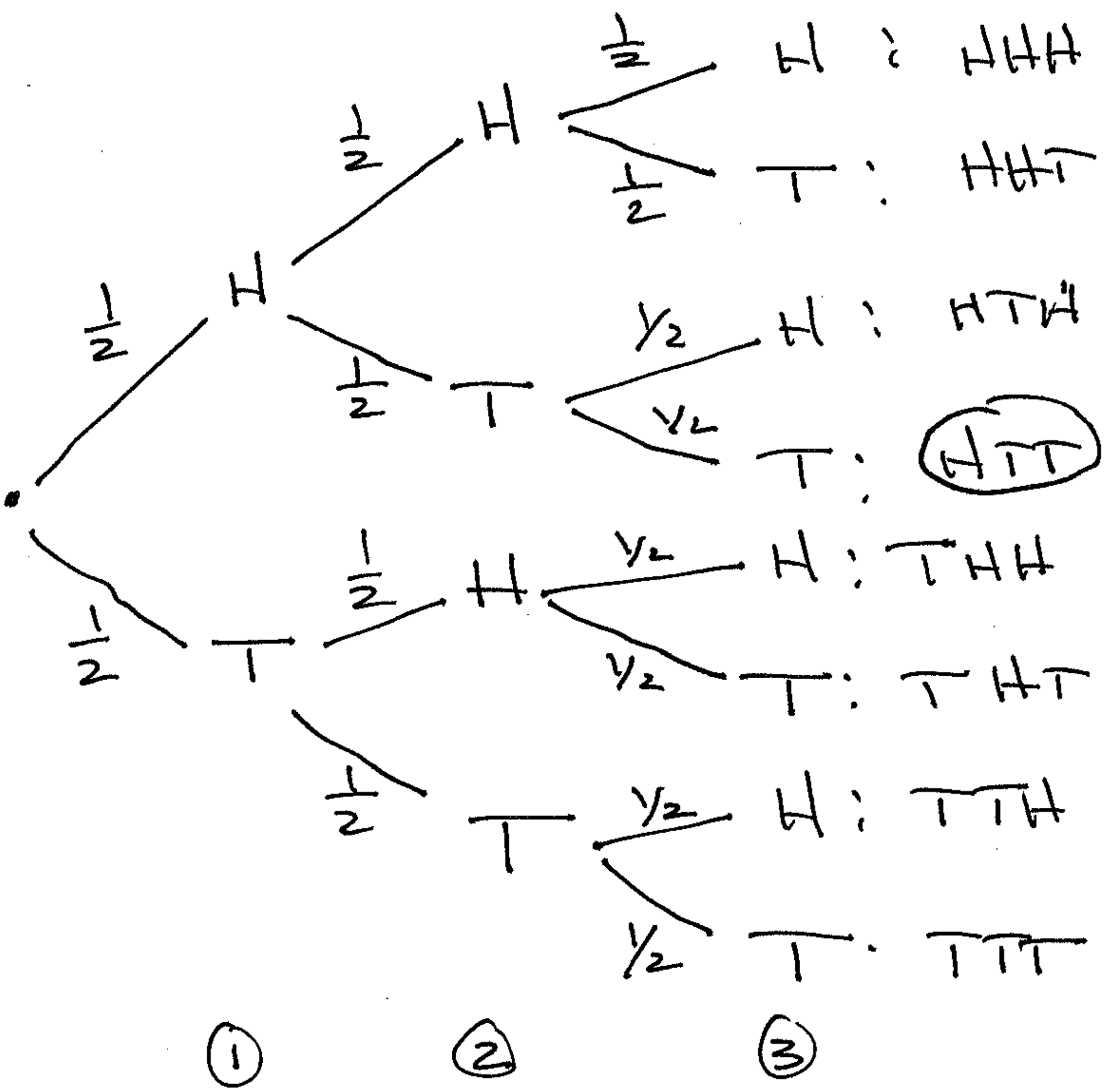
$$P\left(\bigcap_{j=1}^{n} A_i\right) = P(A_1)\cdot P(A_2|A_1)\cdot P(A_3|A_1\cap A_2)$$
$$\cdots P\left(A_n \,\middle|\, \bigcap_{j=1}^{n-1} A_j\right)$$

i.e.

$$P\left(\bigcap_{j=1}^{n} A_i\right) = \prod_{i=1}^{n} P\left(A_i \,\middle|\, \bigcap_{j=1}^{i-1} A_j\right)$$

This rule justifies the practice of multiplying probabilities down branches of a tree diagram.

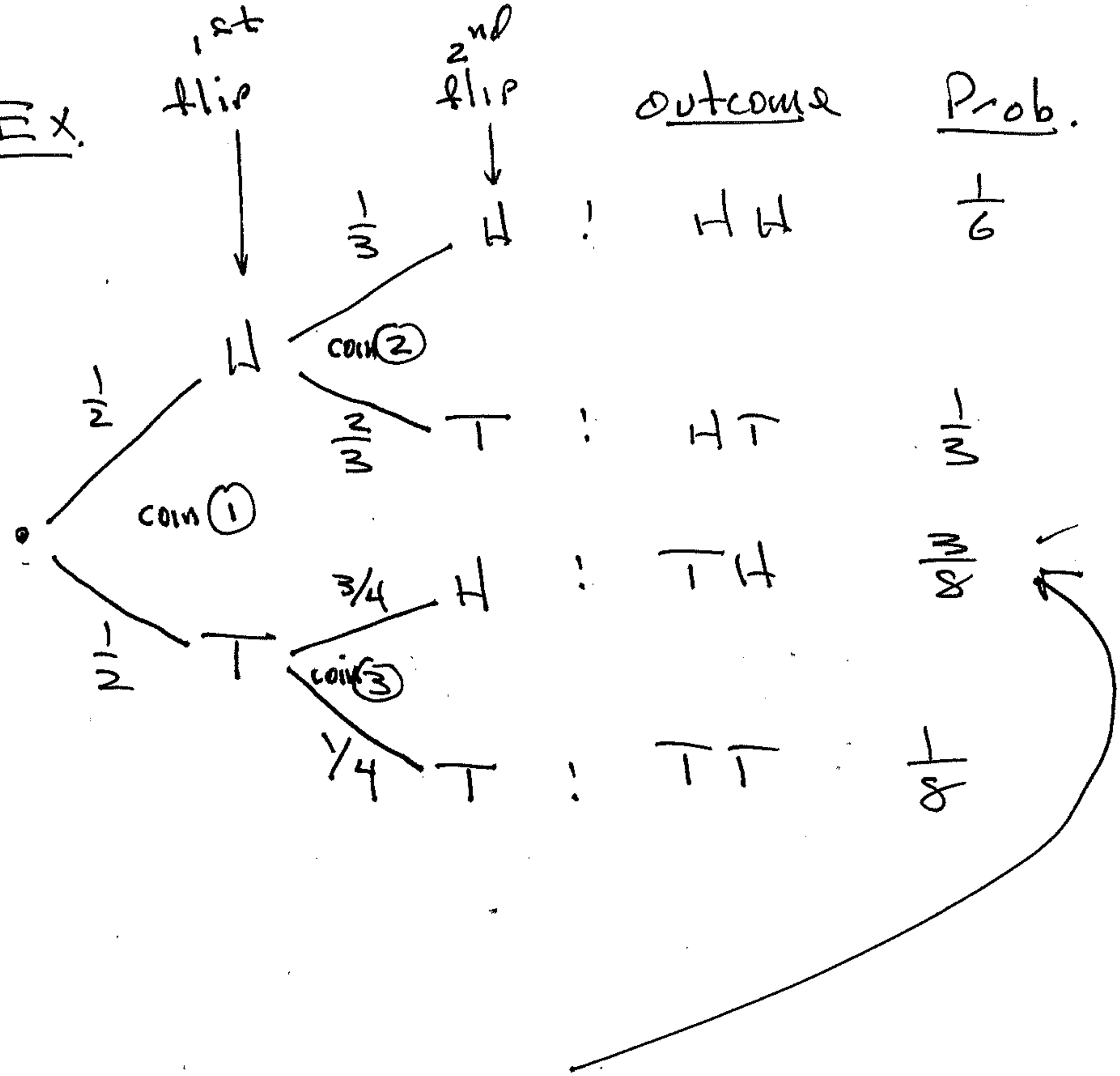
Ex toss 3 fair coins



$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8}$

This also works for loaded coins.

Ex.



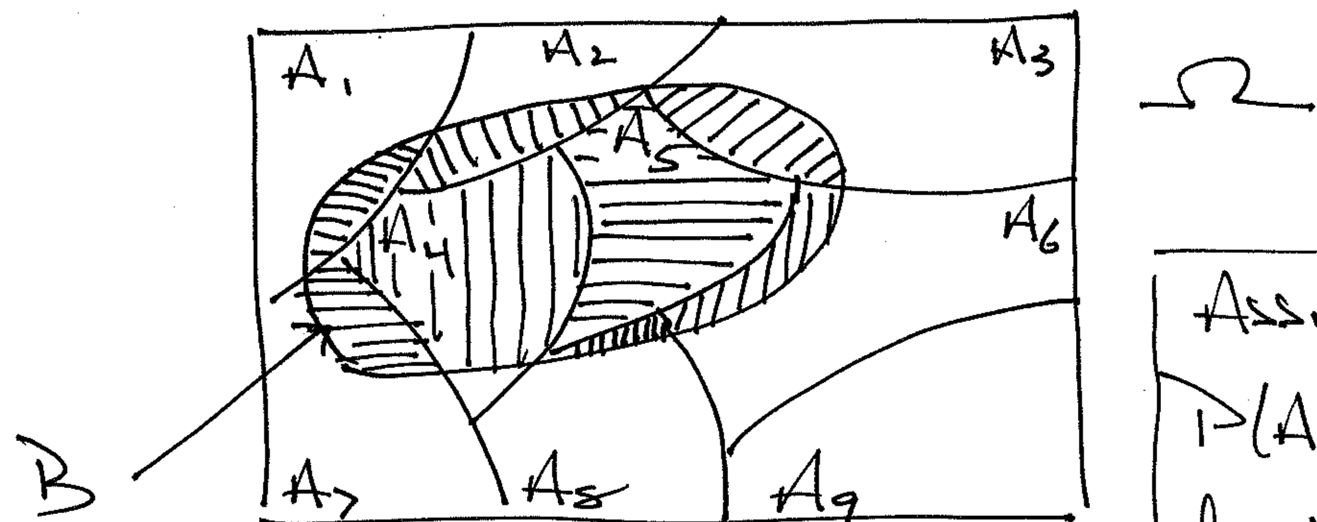
$P(1^{st}\,T \text{ and } 2^{nd}\,H)$

$= P(1^{st}\,T)\cdot P(2^{nd}\,H \mid 1^{st}\,T)$

$= \frac{1}{2}\cdot\frac{3}{4} = \boxed{\frac{3}{8}}$

1.4 Total Probability, Bayes rule

Let $A_1, A_2, \ldots, A_n$ is a partition of Ω:



Assume $P(A_i) > 0$ for $i = 1, \ldots, n$

Total Probability theorem

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \cdots + P(A_n \cap B)$$

$$= P(A_1) \cdot P(B|A_1) + \cdots + P(A_n) P(B|A_n)$$

Ex. Roll a fair 6-sided die. If result is in $\{1, 2, 3\}$, roll again, otherwise stop. What is the probability the the sum (of all rolls) is ≥ 5 ?

Let $A_1 = \{ 1^{st} \text{ roll is } 1\}$ $P(A_1) = \frac{1}{6}$

$A_2 = \{ 1^{st} \text{ roll is } 2\}$ $\frac{1}{6}$

$A_3 = \{ \text{" " " } 3\}$ $\frac{1}{6}$

$A_4 = \{ \text{" " " } 4\}$ $\frac{1}{6}$

$A_5 = \{ \text{" " " } 5\}$ $\frac{1}{6}$

$A_6 = \{ \text{" " " } 6\}$ $\frac{1}{6}$

Let $B = \{ \text{sum} \geq 5 \}$

if A_1 occurs, B occurs iff $2^{nd} \in \{4,5,6\}$

" A_2 " , B iff $2^{nd} \in \{3,4,5,6\}$

" A_3 " , B iff $2^{nd} \in \{2,3,4,5,6\}$

" A_4 " , B does not occur

" A_5 " , B occurs

" A_6 " , B occurs

Thus

$$P(B|A_1) = \frac{3}{6} = \frac{1}{2}$$

$$P(B|A_2) = \frac{4}{6} = \frac{2}{3}$$

$$P(B|A_3) = \frac{5}{6}$$

$$P(B|A_4) = 0$$

$$P(B|A_5) = P(B|A_6) = 1$$

Thus

$$P(B) = P(A_1)\cdot P(B|A_1) + \cdots + P(A_6)\cdot P(B|A_6)$$

$$= \frac{1}{2}\cdot\frac{1}{6} + \frac{2}{3}\cdot\frac{1}{6} + \frac{5}{6}\cdot\frac{1}{6} + 0\cdot\frac{1}{6} + 1\cdot\frac{1}{6} + 1\cdot\frac{1}{6}$$

$$= \frac{1}{6}\left(\frac{1}{2} + \frac{2}{3} + \frac{5}{6} + 0 + 1 + 1\right)$$

$$= \frac{1}{6}\cdot\frac{24}{6} = \boxed{\frac{2}{3}}.$$

<u>Exercise</u> enumerate full sample space, and do it without total Probability.

Baye's Rule

Let $A_1, A_2, \ldots, A_n$ be a Partition of Ω (i.e. Pairwise disjoint, and union $= \Omega$.) Assume $P(A_i) > 0$ for $i = 1, \ldots, n$. Then for each i

$$P(A_i|B) = \frac{P(A_i) \cdot P(B|A_i)}{P(B)}$$

$$= \frac{P(A_i) \cdot P(B|A_i)}{P(A_1) \cdot P(B|A_1) + \cdots + P(A_n) \cdot P(B|A_n)}$$

Proof

recall $P(A \cap B) = P(B) \cdot P(A|B)$

and $P(A \cap B) = P(A) \cdot P(B|A)$

$\therefore P(B) \cdot P(A|B) = P(A) \cdot P(B|A)$

$$\therefore P(A|B) = \frac{P(A) \cdot P(B|A)}{P(B)}$$

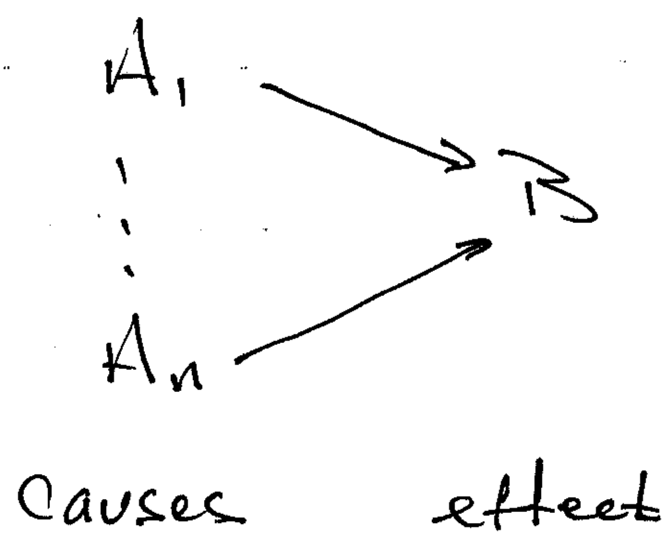
Apply this to A_i, then use total prob. thm on $P(B)$. $\blacksquare$

Bayes' rule is used for inference questions:

causes: $A_1, A_2, \ldots, A_n$

effect: B

we observe effect, and infer causes.



$$A_i \longrightarrow B \; : \; \underset{\substack{\uparrow \\ \left(\text{strength of causation}\right)}}{P(B|A_i)}$$

Causation

$$A_i \longleftarrow B \; : \; \underset{\substack{\uparrow \\ \left(\text{strength of inference}\right)}}{P(A_i|B)}$$

inference

$$\underbrace{P(A_i|B)}_{\substack{\uparrow \\ \text{Posterior}}} = \frac{\overbrace{P(A_i)}^{\text{Prior}} \cdot \overbrace{P(B|A_i)}^{\text{likelihood}}}{\underbrace{P(B)}_{\substack{\uparrow \\ \text{marginal}}}}$$

Ex. 'Disease' example again.

$T = \{\text{test positive}\}$

$D = \{\text{has disease}\}$

$P(T|D) = .99$

$P(T|D^c) = .10$

$P(D) = .05, \; P(D^c) = .95$

find $P(\text{disease}|\text{test})$, i.e.

$$P(D|T) = \frac{P(D)\cdot P(T|D)}{P(T)}$$

$$= \frac{P(D)\cdot P(T|D)}{P(D)\cdot P(T|D) + P(D^c)\cdot P(T|D^c)}$$

$$= \frac{(.05)(.99)}{(.05)(.99) + (.95)(.10)}$$

$$= \boxed{.3426}$$