CSE 107 10-29-24

Ex. (Piecewise constant PDF)

Alice's driving time to work is 40-44 minutes in good weather, and 44-50 minutes in bad, uniformly distributed in each case. Suppose

$$P(\text{good weather}) = \frac{2}{3}$$

Determine the PDF of X, Alice's driving time.

<u>note</u>: range$(X) = [40, 50]$.

we have

$$f_X(x) = \begin{cases} c_1 & \text{if } 40 \le x \le 44 \text{ (good)} \\ c_2 & \text{if } 44 < x \le 50 \text{ (bad)} \\ 0 & \text{otherwise} \end{cases}$$

Also

$$\frac{2}{3} = P(\text{good}) = P(40 \le X \le 44)$$

$$= \int_{40}^{44} f_X(x)\,dx = \int_{40}^{44} c_1\,dx = c_1 \cdot 4$$

$$\therefore \quad c_1 = \frac{1}{\cancel{4}_{2}} \cdot \frac{\cancel{2}}{3} = \boxed{\frac{1}{6}}$$

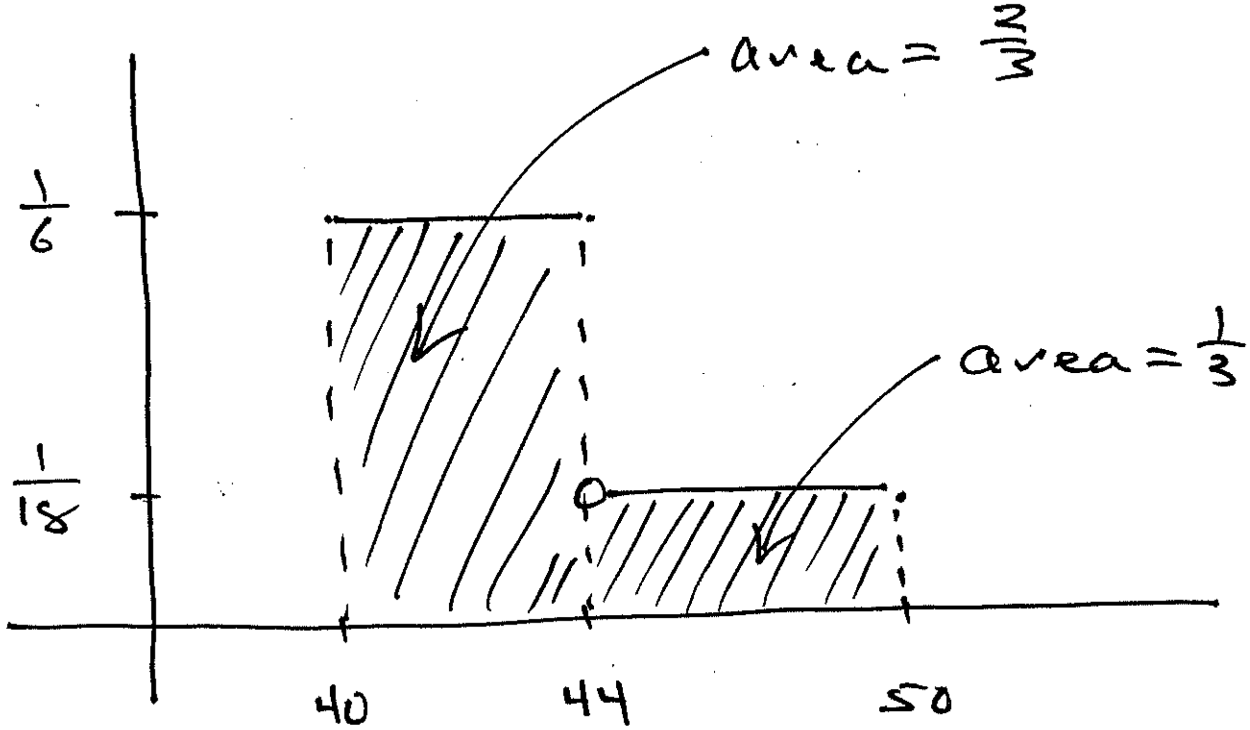
and

$$\frac{1}{3} = P(\text{bad}) = P(44 \le X \le 50)$$

$$= \int_{44}^{50} c_2\,dx = 6 \cdot c_2$$

$$\therefore\ c_2 = \frac{1}{6} \cdot \frac{1}{3} = \boxed{\frac{1}{18}}$$

$$\therefore\ f_X(x) = \begin{cases} \frac{1}{6} & 40 \le x \le 44 \\ \frac{1}{18} & 44 < x \le 50 \\ 0 & \text{otherwise} \end{cases}$$



<u>note</u>: a PDF can become infinite.

<u>Ex</u>

$$f_X(x) = \begin{cases} \dfrac{1}{2\sqrt{x}} & \text{if } 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^1 \frac{1}{2} x^{-\frac{1}{2}}\,dx$$

$$= \lim_{a \to 0^+} \int_a^1 \frac{1}{2} x^{-\frac{1}{2}}\,dx$$

$$= \lim_{a \to 0^+} \cancel{\frac{1}{2}} \cdot \frac{x^{\frac{1}{2}}}{\cancel{\frac{1}{2}}} \Bigg|_a^1$$

$$= \lim_{a \to 0^+} \left(1 - \sqrt{a}\right) = \boxed{1}$$

Defn

The expected value of a continuous r.v. X is

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

Expected value rule:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

see Prob. #4 on P. 185 with soln.

The n^{th} moment of X is

$$E[X^n]$$

and

$$Var(X) = E[(X - E[X])^2]$$

Thm: $Var(X) = E[X^2] - E[X]^2$

exercise: Prove this.

If $Y = aX + b$ then

$$E[Y] = aE[X] + b$$

and

$$Var(Y) = a^2 Var(X)$$

exercise: Prove these

<u>Ex.</u> cont. uniform r.v. on $[a,b]$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

so

$$E[X] = \int_a^b x\left(\frac{1}{b-a}\right)dx$$

$$= \frac{1}{b-a} \cdot \frac{x^2}{2}\Big|_a^b = \frac{1}{2} \cdot \frac{\overbrace{b^2-a^2}^{(b-\cancel{a})(b+a)}}{(\cancel{b-a})}$$

$$= \frac{a+b}{2}$$

$$E[X^2] = \underset{\substack{\uparrow \\ \text{exercise}}}{\cdots\cdots} = \frac{a^2+ab+b^2}{3}$$

Hence

$$\mathrm{Var}(X) = \ldots\ldots = \frac{(b-a)^2}{12}$$

exercise

<u>Exercise</u>

Alice driving example

$$f_X(x) = \begin{cases} 1/6 & 40 \le x \le 44 \\ 1/18 & 44 < x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

find:

$E[X] =$

$E[X^2] =$

$\mathrm{Var}(X) =$

Ex. Exponential Random Variable

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Parameter: $\lambda > 0$

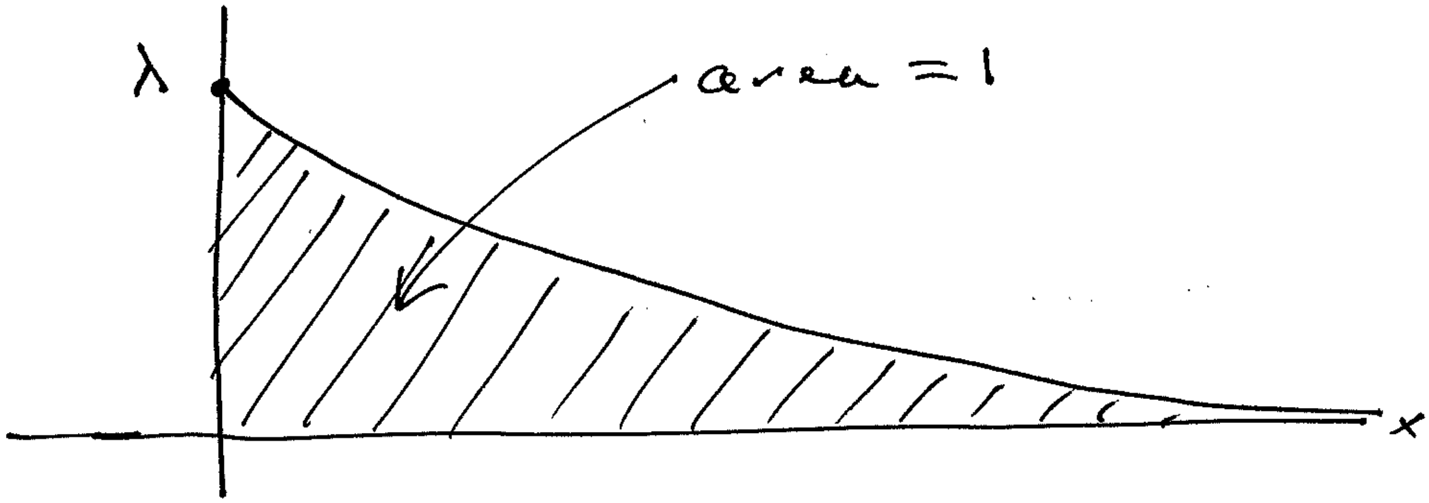
This is the continuous time analog of the geometric r.v.

check

$$\int_0^\infty \lambda e^{-\lambda x} dx = \lambda \cdot \left(\frac{1}{-\lambda}\right) \cdot e^{-\lambda x} \Big|_0^\infty$$

$$= -(0-1) = \boxed{1}$$

graph



Exercise: show that

- $E[X] = \frac{1}{\lambda}$ ✓

and

- $Var(X) = \frac{1}{\lambda^2}$

Hint: use integration by parts.

$$E[X] = -\int_0^\infty x \cdot \left(-\lambda e^{-\lambda x}\right) dx$$

$$\begin{cases} u = x & dv = -\lambda e^{-\lambda x} \\ du = dx & v = e^{-\lambda x} \end{cases}$$

$$= -\left(x e^{-\lambda x} \Big|_0^\infty - \int_0^\infty e^{-\lambda x}\, dx \right)$$

$$= -\left(\cancel{(0 - 0)} - \left(-\frac{1}{\lambda}\right) e^{-\lambda x} \Big|_0^\infty \right)$$

$$= -\frac{1}{\lambda}(0 - 1) = \frac{1}{\lambda} \quad \checkmark$$

<u>Note</u>: for X exponential, Param. λ

$E[X]$ has units $\frac{\text{time}}{\text{event}}$

$\frac{1}{\lambda}$ " " $\frac{\text{time}}{\text{event}}$

λ " " $\frac{\text{events}}{\text{time}}$

Thus we call λ the 'event rate' or 'arrival rate' or 'rate' of the r.v. X.

<u>Theorem</u>

If X is Exponential then

$$P(X \geq a) = \begin{cases} e^{-\lambda a} & \text{if } a > 0 \\ 1 & \text{if } a \leq 0 \end{cases}$$

<u>Proof</u>

let $a > 0$. Then

$$P(X \geq a) = \int_a^\infty \lambda e^{-\lambda x}\,dx$$

$$= \cancel{\lambda}\left(\frac{1}{-\cancel{\lambda}}\right) e^{-\lambda x}\Big|_a^\infty$$

$$= -\left(0 - e^{-\lambda a}\right) = e^{-\lambda a}. \quad \blacksquare$$

Ex.

The time X until the next earthquake (of 4.0 or above) is modeled as an exp. r.v. with a mean of 53 minutes (worldwide)

(a)
What is the probability that such an earthquake occurs in next 10 minutes?

$$E[X] = \frac{1}{\lambda} = 53 \Rightarrow \lambda = \frac{1}{53}$$

hence

$$P(0 \leq X \leq 10) = P(X \geq 0) - P(X > 10)$$

$$= 1 - e^{-\lambda 10} = 1 - e^{-\frac{10}{53}} = \boxed{0.1719}$$

(b) exercise

determine a number a such that the prob. of an earthquake within next a minutes is 0.99, i.e. find a such that

$$P(0 \le X \le a) = .99$$

answer:

$$a = 53 \ln(100)$$

$$= 244.07 \text{ min}$$

$$= 4.068 \text{ hours}.$$

3.2 Cumulative Distribution Functions

<u>Defn</u>

Let X be a random variable (disc. or cont.) The <u>cummulative distribution function</u> (CDF) of X is

$$F_X(x) = P(X \leq x)$$

$$F_X(x) = \begin{cases} \sum_{k \le x} P_X(k) & (X \text{ discrete}) \\ \int_{-\infty}^{x} f_X(t)\, dt & (X \text{ continuous}) \end{cases}$$

Ex. (discrete)

PMF : $P_X(k) = \begin{cases} 3/10 & k=1 \\ 5/10 & k=2 \\ 2/10 & k=3 \\ 0 & \text{otherwise} \end{cases}$

⋮