CSE 107 10-18-24

Supplemental

Recall

$$\boxed{P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}}$$

Thus

$$\boxed{P_{X,Y}(x,y) = P_Y(y) \cdot P_{X|Y}(x|y)}$$

also

$$\boxed{P_{X,Y}(x,y) = P_X(x) \cdot P_{Y|X}(y|x)} \quad *$$

Ex. Professor Problem

A professor answers questions wrongly with prob. $1/4$, independent of other questions. The number of questions asked is 0, 1 or 2, with equal probability. Let

$$X = \#\text{ questions asked}$$

and

$$Y = \#\text{ wrong answers}$$

(a) Determine the joint PMF

$p_{X,Y}(x,y)$.

Note: $$p_X(x) = \begin{cases} 1/3 & \text{if } x = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

$P_X(x)$

x	0	1	2
$P_X(x)$	1/3	1/3	1/3

$P_{Y|X}(y|x)$

y \ x	0	1	2
2			1/16
1		1/4	3/8
0	1	3/4	9/16

$P_{X,Y}(x,y)$ and $P_Y(y)$

y \ x	0	1	2	→	$P_Y(y)$
2			1/48	→	1/48
1		1/12	1/8	→	10/48
0	1/3	1/4	3/16	→	37/48

(b) Find $P_Y(y)$.

(c) Find $P_{X|Y}(x|y) = \dfrac{P_{X,Y}(x,y)}{P_Y(x)}$

answer

$P_{X|Y}(x|y)$

y \ x	0	1	2
2			1
1		$2/5$	$3/5$
0	$16/37$	$12/37$	$9/37$

Conditional Expectation

Let X be a r.v., $A \subseteq \Omega$ with $P(A) > 0$.

Define

$$\boxed{E[X|A] = \sum_x x \, P_{X|A}(x)}$$

also

$$\boxed{E[g(X)|A] = \sum_x g(x) \, P_{X|A}(x)}$$

we can condition on $A=\{Y=y\}$ where Y is a r.v. and $y \in \text{range}(Y)$

$$\boxed{E[X \mid Y=y] = \sum_x x \, P_{X|Y}(x|y)}$$

<u>Theorem</u> (Total Expectation Thm)

Let $A_1, A_2, \ldots, A_n \subseteq \Omega$ form a partition of Ω. Then

$$\boxed{E[X] = \sum_{i=1}^{n} P(A_i) \cdot E[X \mid A_i]}$$

<u>Proof</u>. By total Prob. thm

$$P_X(x) = \sum_{i=1}^{n} P(A_i) \cdot P_{X|A_i}(x)$$

multiply both sides by x and sum over $x \in \text{range}(X)$.

$$E[X] = \sum_x x \, p_X(x)$$

$$= \sum_x x \cdot \sum_{i=1}^{n} P(A_i) \cdot p_{X|A_i}(x)$$

$$= \sum_x \sum_{i=1}^{n} P(A_i) \cdot x \, p_{X|A_i}(x)$$

$$= \sum_{i=1}^{n} P(A_i) \cdot \sum_x x \, p_{X|A_i}(x)$$

$$= \sum_{i=1}^{n} P(A_i) \cdot E[X|A_i]$$

■

see summary on p. 10.4 of Text.

Mean and Variance of a Geometric r.v.

Recall, if X is geom. with Param. p:

$$P_X(k) = (1-p)^{k-1} \cdot p \qquad (k=1,2,3,\ldots)$$

so

$$E[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1} p$$

and

$$E[X^2] = \sum_{k=1}^{\infty} k^2 (1-p)^{k-1} p$$

and

$$\mathrm{Var}(X) = E[X^2] - E[X]^2$$

We use total Expectation thm.

Let

$$A_1 = \{X = 1\} = \{1^{st} \text{ flip is heads}\}$$

and

$$A_2 = \{X > 1\} = \{1^{st} \text{ flip is tails}\}.$$

Then $P(A_1) = p$ and $P(A_2) = 1 - p$

Note $E[X \mid A_1] = 1$. Also note:

$$\text{seq. of flips:} \quad \begin{array}{ccccc} \frac{H}{T} & \frac{H}{T} & \frac{H}{T} & \frac{H}{T} & \frac{H}{T} \\ 1 & 2 & 3 & 4 & 5 \; \ldots \end{array}$$

$$\underbrace{2 \quad 3 \quad 4 \quad 5 \; \ldots}_{\text{identical seq.}}$$

$$E[X \mid A_2] = E[1 + X] = 1 + E[X]$$

Now by total expectation thm:

$$E[X] = p \cdot E[X|A_1] + (1-p) \cdot E[X|A_2]$$

$$= p \cdot 1 + (1-p)\left(1 + E[X]\right)$$

$$\cancel{E[X]} = \cancel{p} + 1 + \cancel{E[X]} - \cancel{p} - pE[X]$$

$$\therefore \quad pE[X] = 1$$

$$\therefore \quad E[X] = \boxed{\frac{1}{p}}$$

similarly

$$E[X^2 | A_1] = 1$$

and

$$E[X^2 | A_2] = E[(1+X)^2]$$

$$= E[1 + 2X + X^2]$$

$$= 1 + 2E[X] + E[X^2]$$

Therefore

$$E[X^2] = p\cdot 1 + (1-p)\left(1 + 2\overbrace{E[X]}^{\frac{1}{p}} + E[X^2]\right)$$

$$E[\cancel{X^2}] = \cancel{p} + (1-\cancel{p}) + (1-p)\cdot\frac{2}{p} + (\cancel{1}-p)\,E[X^2]$$

$$p\,E[X^2] = 1 + \frac{2-2p}{p} = \frac{p-2p+2}{p} = \frac{2-p}{p}$$

$$\therefore\ E[X^2] = \frac{2-p}{p^2} = \frac{2}{p^2} - \frac{1}{p}$$

Finally

$$Var(X) = \left(\frac{2}{p^2} - \frac{1}{p}\right) - \left(\frac{1}{p}\right)^2 = \frac{1}{p^2} - \frac{1}{p}$$

$$Var(X) = \boxed{\frac{1-p}{p^2}}$$ ✓