cse 107 10-17-24

- midterm 1 : Thur. 10/24

: continue

<u>more than 2 r.v.s</u>

Suppose X, Y, Z are random variables,

<u>Defn</u>

The joint PMF of X, Y, Z is

$$P_{X,Y,Z}(x,y,z) = P(X=x, Y=y, Z=z)$$

Then

$$P_{X,Y}(x,y) = \sum_{z} P_{X,Y,Z}(x,y,z)$$

$$P_{X,Z}(x,z) = \sum_{y} P_{X,Y,Z}(x,y,z)$$

$$P_{Y,Z}(y,z) = \sum_{x} P_{X,Y,Z}(x,y,z)$$

Expected value rule:

$$E[g(X,Y,Z)] = \sum_{(x,y,z)} g(x,y,z) P_{X,Y,Z}(x,y,z)$$

Suppose we have n random variables

$$X_1, X_2, X_3, \ldots, X_n$$

Then

$$E[a_1X_1 + a_2X_2 + \cdots + a_nX_n + b]$$

$$= a_1E[X_1] + a_2E[X_2] + \cdots + a_nE[X_n] + b$$

by the expected value rule.

The mean of the Binomial

let X be a Binomial r.v. with parameters n, p. so

$$P_X(k) = \binom{n}{k} p^k \cdot (1-p)^{n-k} \quad (0 \leq k \leq n)$$

so

$$E[X] = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

we have a better method for computing $E[X]$.

Note

$$X = X_1 + X_2 + X_3 + \cdots + X_n$$

where each X_i is Bernoulli with parameter p. Recall

$$E[X_i] = 1 \cdot p + 0 \cdot (1-p) = p$$

so

$$E[X] = E[X_1 + \cdots + X_n]$$

$$= E[X_1] + \cdots + E[X_n]$$

$$= p + p + \cdots p$$

$$= np$$

we'll see later that

$$Var(X) = np(1-p)$$

Ex. The hat problem.

suppose n people throw their hats into a box, then each picks a hat at random. This implies that each assignment

$$\{hats\} \longrightarrow \{people\}$$

is equally likely.

Thus each person has prob. $\frac{1}{n}$ of getting their own hat back. Let

$$X = \#\text{ people who get their own hat back}$$

Find $E[X]$. Observe

$$X = X_1 + X_2 + \cdots + X_n$$

where each X_i is Bernoulli with param. $p = \frac{1}{n}$

$$X_i = \begin{cases} 1 & \text{if } i^{th} \text{ person hat back} \\ 0 & \text{otherwise} \end{cases}$$

we have

$$E[X_i] = p = \frac{1}{n}$$

and

$$E[X] = E[X_1] + \cdots + E[X_n]$$

$$= \frac{1}{n} + \cdots + \frac{1}{n}$$

$$= n \cdot \frac{1}{n} = \boxed{1}$$

2.6 Conditioning

Given a r.v.

$$X : \Omega \to \mathbb{R}$$

any valid prob. law $\mathbb{P} : \mathbb{P}(\Omega) \to [0,1]$ can be used to define

$$P_X(x) = \mathbb{P}(X = x)$$

This includes a conditional probability law $\mathbb{P}(\cdot \mid A)$ where $A \subseteq \Omega$ and $\mathbb{P}(A) > 0$.

Defn

let X, A be as above. The <u>conditional PMF of X given A</u>

is

$$p_{X|A}(x) = P(\{X=x\} \mid A)$$

$$= \frac{P(\{X=x\} \cap A)}{P(A)}$$

<u>note</u>: the events

$$\{X=x\} \cap A$$

for $x \in \text{range}(X)$, form a partition of A

so by the total Prob. thm.

$$P(A) = \sum_x P(\{X=x\} \cap A)$$

Thus

$$\sum_x P_{X|A}(x) = \sum_x \frac{P(\{X=x\} \cap A)}{P(A)}$$

$$= \frac{1}{P(A)} \cdot \sum_x P(\{X=x\} \cap A)$$

$$= \frac{P(A)}{P(A)} = 1 .$$

so $P_{X|A}(x)$ is a valid PMF.

Ex. roll 2 independent 6-sided dice, let X be the sum.

$$P_X(k) = \begin{cases} 1/36 & \text{if } k = 2, 12 \\ 2/36 & k = 3, 11 \\ 3/36 & k = 4, 10 \\ 4/36 & k = 5, 9 \\ 5/36 & k = 6, 8 \\ 6/36 & k = 7 \end{cases}$$

$X =$ 2 3 4 5 6 7

11	12	13	14	15	16	
21	22	23	24	25	26	8
31	32	33	34	35	36	9
41	42	43	44	45	46	10
51	52	53	54	55	56	11
61	62	63	64	65	66	12

A

Let $A = \{\min \geq 3\}$, $P(A) = \frac{16}{36}$

so

$$P_{X|A}(k) = \begin{cases} 1/16 & \text{if } k = 6, 12 \\ 2/16 & k = 7, 11 \\ 3/16 & k = 8, 10 \\ 4/16 & k = 9 \\ 0 & \text{otherwise} \end{cases}$$

$$P_{X|A}(6) = \frac{P(\{X=6\} \cap A)}{P(A)} = \frac{1/36}{16/36} = \frac{1}{16}$$

Ex. flip a coin with $P(H)=p$ repeatedly until 1^{st} heads. Let X be # flips. Then X is geometric with param. p

$$p_X(k) = \begin{cases} (1-p)^{k-1} \cdot p & \text{if } k = 1,2,3,\ldots \\ 0 & \text{otherwise} \end{cases}$$

Let

$A = \{1^{st}$ head occurs within n flips$\}$

Then

$$P(A) = \sum_{k=1}^{n} p_X(k) = \sum_{k=1}^{n} (1-p)^{k-1} \cdot p$$

$$= p \cdot \sum_{k=1}^{n} (1-p)^{k-1}$$

$$= p \cdot \sum_{k=0}^{n-1} (1-p)^{k} \qquad \left\{ \begin{array}{l} k \leftarrow k+1 \\ k+1 = 1 \Rightarrow k = 0 \\ k+1 = n \Rightarrow k = n-1 \end{array} \right.$$

$$= \cancel{p} \cdot \frac{1-(1-p)^n}{\cancel{1-(1-p)}}$$

$$= 1-(1-p)^n .$$

another way

$$P(A) = 1 - P(A^c)$$

$$= 1 - P(1^{st}\ n \text{ flips are tails})$$

$$= 1 - (1-p)^n$$

Thus

$$P_{X|A}(k) = \begin{cases} \dfrac{(1-p)^{k-1} \cdot p}{1-(1-p)^n} & \text{if } k = 1, \dots, n \\ 0 & \text{otherwise} \end{cases}$$

<u>note</u>: $\{X=k\} \cap A = \{X=k\}$

$$\therefore P(\{X=k\} \cap A) = P(X=k) = (1-p)^{k-1} \cdot p$$

<u>Exercise</u>

check: $\displaystyle\sum_{k=1}^{n} P_{X|A}(k) = 1$

Now let Y be another r.v.

with

$$P_Y(y) = P(Y=y) > 0,$$

for all $y \in \text{range}(Y)$. Specialize to case $A = \{Y=y\}$

<u>Defn</u>

The cond. PMF of X given Y

$$P_{X|Y}(x|y) = P(X=x \mid Y=y)$$

$$= \frac{P(X=x, Y=y)}{P(Y=y)}.$$