cse 107 10-15-24

Rmks on lab 2

Ex.

~~A~~ ~~A~~ ~~A~~

~~C~~ ~~C~~ ~~C~~ ~~C~~ ~~C~~

$a = 3$

$c = 5$

1. start : ~~C~~ ~~C~~ A replace
2. restart : ~~A~~ ~~A~~ C replace
3. restart : ~~C~~ A replace
4. restart : ~~C~~ ~~C~~ A replace
5. restart : ~~A~~

last removed : <u>Azure</u>

Ex The Poisson r.v. with Param: $\lambda > 0$

$$P_X(K) = e^{-\lambda} \cdot \frac{\lambda^K}{K!} \quad (K = 0, 1, 2, \ldots)$$

we will show

(1) $E[X] = \lambda$

(2) $E[X^2] = \lambda^2 + \lambda$

Proof of (1):

$$E[X] = \sum_{K=\cancel{0}1}^{\infty} K \cdot e^{-\lambda} \frac{\lambda^K}{K!}$$

$$= e^{-\lambda} \cdot \sum_{K=1}^{\infty} K \cdot \frac{\lambda^K}{K!} \qquad (*)$$

$$= \lambda \cdot e^{-\lambda} \cdot \sum_{K=1}^{\infty} \frac{\lambda^{K-1}}{(K-1)!}$$

$$= \lambda e^{-\lambda} \cdot \sum_{K=0}^{\infty} \frac{\lambda^{K}}{K!} \quad \left\{ \begin{array}{l} \text{replace} \\ K \leftarrow K+1 \end{array} \right.$$

$$= \lambda \cdot e^{-\lambda} \cdot e^{\lambda} = \lambda \qquad \blacksquare$$

<u>Proof of (2)</u>

$$E[X^2] = \sum_{K=\cancel{0}\,1}^{\infty} K^2 \cdot e^{-\lambda} \frac{\lambda^K}{K!}$$

$$= \lambda e^{-\lambda} \cdot \sum_{K=1}^{\infty} K \cdot \frac{\lambda^{K-1}}{(K-1)!}$$

$$= \lambda e^{-\lambda} \cdot \sum_{K=0}^{\infty} (K+1) \frac{\lambda^{K}}{K!} \quad \left\{ \begin{array}{l} \text{replace} \\ K \leftarrow K+1 \end{array} \right.$$

$$= \lambda e^{-\lambda} \left(\sum_{K=\cancel{0}\,1}^{\infty} K \cdot \frac{\lambda^K}{K!} + \sum_{K=0}^{\infty} \frac{\lambda^K}{K!} \right)$$

$$= \lambda \left(e^{-\lambda} \cdot \sum_{k=1}^{\infty} k \cdot \frac{\lambda^k}{k!} + e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right)$$

$$= \lambda \left(\lambda + e^{-\lambda} \cdot e^{\lambda} \right) = \lambda(\lambda + 1)$$

$$= \lambda^2 + \lambda \qquad \blacksquare$$

Now

$$\mathrm{Var}(X) = E[X^2] - E[X]^2$$

$$= (\lambda^2 + \lambda) - \lambda^2 = \lambda$$

Ex. The Quiz Problem.

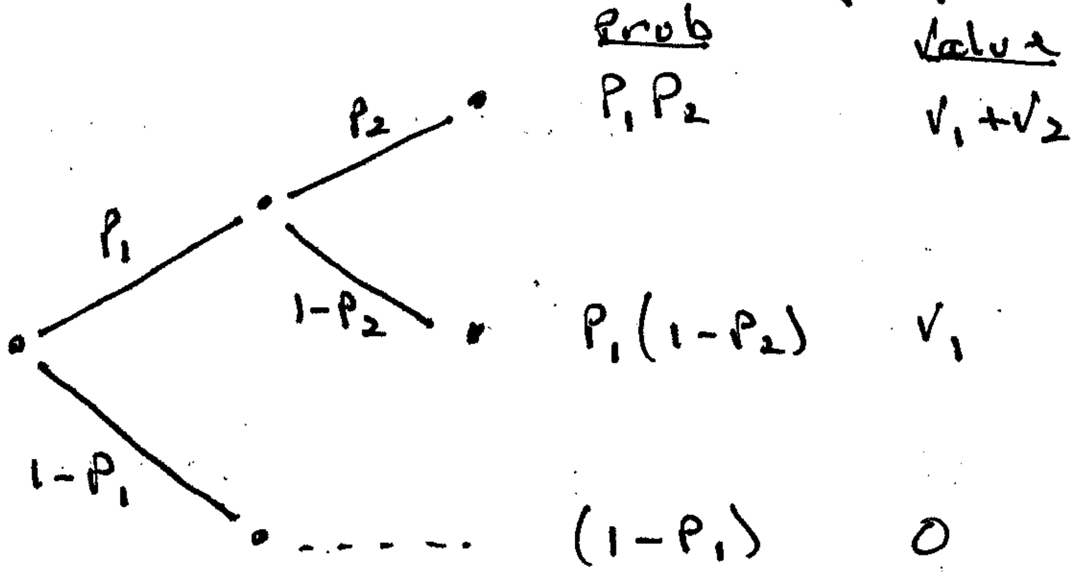
A game consists of 2 questions

	Q_1	Q_2
P(correct)	p_1	p_2
Payoff \$	v_1	v_2

The 1st wrong answer ends the game.
You can choose order: (Q_1, Q_2) or (Q_2, Q_1)

Let X be total payoff for order (Q_1, Q_2).

- find PMF of X



so

$$P_X(x) = \begin{cases} p_1 p_2 & x = v_1 + v_2 \\ p_1(1-p_2) & x = v_1 \\ (1-p_1) & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$E[X] = (v_1+v_2)\, p_1 p_2 + v_1 p_1 (1-p_2)$$

$$= v_1 p_1 + v_2 \cdot p_1 p_2$$

- Let Y be total payoff for order (Q_2, Q_1)

By symmetry

$$E[Y] = V_2 P_2 + V_1 P_1 P_2$$

- When should we choose (Q_1, Q_2)?

ans. $E[X] > E[Y]$

$$V_1 P_1 + V_2 P_1 P_2 > V_2 P_2 + V_1 P_1 P_2$$

$$\vdots$$

$$\frac{V_1 P_1}{1 - P_1} > \frac{V_2 P_2}{1 - P_2}$$

Exercise

analyse this problem for 3 questions: Q_1, Q_2, Q_3

which of $6 = 3!$ orders is best.

Conjecture: sort by $\mathrm{Val}\frac{P(\text{right})}{P(\text{wrong})}$.

2.5 Joint PMFs, Multiple r.v.s

Defn

Given two r.v.s X, Y their **joint PMF** is

$$p_{X,Y}(x,y) = P(X=x, Y=y)$$

$$= P(\{X=x\} \cap \{Y=y\})$$

The joint ~~PMF~~ PMF allows us to compute probabilities assoc. with both X and Y.

For instance, if

$$S \subseteq \text{range}(X) \times \text{range}(Y) \subseteq \mathbb{R} \times \mathbb{R}$$

then

$$P\big((X,Y) \in S\big) = \sum_{(x,y) \in S} p_{X,Y}(x,y)$$

<u>EX.</u> Suppose $\text{range}(X) = \{1,2,3,4\}$ and $\text{range}(Y) = \{1,2,3\}$. Let the joint PMF be

		$x=1$	$x=2$	$x=3$	$x=4$	$p_Y(y)$
$p_X(x)$		$4/20$	$5/20$	$6/20$	$5/20$	
y	3	$1/20$	$1/20$	$2/20$	$1/20$	$5/20$
	2	$2/20$	$3/20$	$3/20$	$1/20$	$9/20$
	1	$1/20$	$1/20$	$1/20$	$3/20$	$6/20$

$p_{X,Y}(x,y)$

Suppose $Y=3$. Since

$$\{X=1\}\cup\{X=2\}\cup\{X=3\}\cup\{X=4\}=\Omega$$

By the total Prob. thm

$$P(Y=3)=P(X=1,Y=3)+P(X=2,Y=3)+P(X=3,Y=3)$$
$$+P(X=4,Y=3)$$

So we can compute $P_X(x)$ and $P_Y(y)$

By

$$\boxed{P_X(x) = \sum_y P_{X,Y}(x,y)}$$

marginal PMF

and

$$\boxed{P_Y(y) = \sum_x P_{X,Y}(x,y)}$$

marginal PMF

Functions of multiple r.v.s

let $Z = g(X, Y)$, where

$$g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$$

Then

$$p_Z(z) = \sum_{\{(x,y) \mid g(x,y) = z\}} p_{X,Y}(x, y)$$

Expected Value rule:

$$\boxed{E[g(X,Y)] = \sum_x \sum_y g(x,y) \cdot p_{X,Y}(x,y)}$$

Exercise Prove this.

Ex. let $Z = aX + bY + c$

$$E[aX + bY + c] = aE[X] + bE[Y] + c$$ ✓

Proof

$$E[aX+bY+c] = \sum_x \sum_y (ax+by+c) P_{X,Y}(x,y)$$

$$= a \sum_x \sum_y x P_{X,Y}(x,y) + b \sum_x \sum_y y P_{X,Y}(x,y)$$

$$+ c \sum_x \sum_y P_{X,Y}(x,y)$$

$$= a \sum_x x P_X(x) + b \sum_y y P_Y(y) + c$$