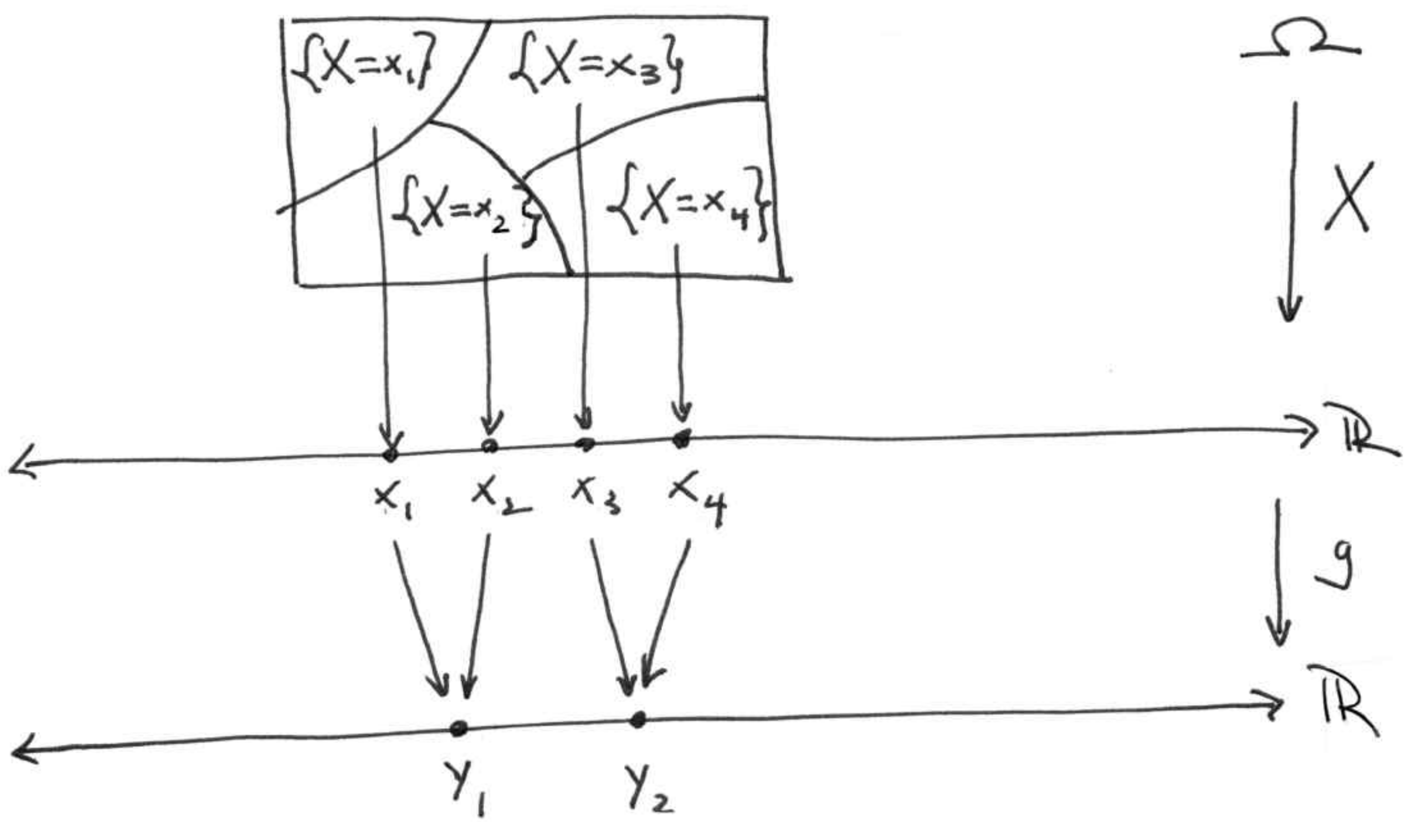
CSE 107 10-11-24

Supplemental

Expected Value rule

$$E[g(X)] = \sum_x g(x) \cdot P_X(x)$$

Picture/example



$$E[g(X)] = E[Y]$$

$$= y_1 \cdot p_Y(y_1) + y_2 p_Y(y_2)$$

$$= y_1 \cdot (p_X(x_1) + p_X(x_2)) + y_2 (p_X(x_3) + p_X(x_4))$$

$$= y_1 p_X(x_1) + y_1 p_X(x_2) + y_2 p_X(x_3) + y_2 p_X(x_4)$$

$$= g(x_1) p_X(x_1) + g(x_2) p_X(x_2) + g(x_3) p_X(x_3) + g(x_4) p_X(x_4)$$

$$= \sum_x g(x) \cdot p_X(x)$$

Proof of Expected Value rule

$$\{Y = y\} = \{\omega \mid Y(\omega) = y\}$$

$$= \{\omega \mid g(X(\omega)) = y\}$$

$$= \bigcup_{\{x \mid g(x)=y\}} \{\omega \mid X(\omega) = x\}$$

$$= \bigcup_{\{x \mid g(x)=y\}} \{X = x\} \quad \left\{ \begin{array}{l} \text{disjoint} \\ \text{union} \end{array} \right.$$

$$\therefore \; P_Y(y) = P(Y=y)$$

$$= \sum_{\{x \mid g(x)=y\}} P(X=x)$$

$$= \sum_{\{x \mid g(x)=y\}} P_X(x)$$

Now

$$E[Y] = \sum_y y \, P_Y(y)$$

$$= \sum_{y} y \cdot \sum_{\{x \mid g(x) = y\}} p_X(x)$$

$$= \sum_{y} \sum_{\{x \mid g(x) = y\}} y \cdot p_X(x)$$

$$= \sum_{y} \sum_{\{x \mid g(x) = y\}} g(x) \cdot p_X(x)$$

$$= \sum_{x} g(x) p_X(x)$$

$\blacksquare$

Ex. $Y = aX + b \quad (a, b \in \mathbb{R})$.

$$E[Y] = E[aX + b]$$

$$= \sum_{x} (ax + b) p_X(x)$$

$$= \sum_x \left(a x p_X(x) + b p_X(x) \right)$$

$$= a \sum_x x p_X(x) + b \sum_x p_X(x)$$

$$= a E[X] + b$$

In Particular:

- $E[aX] = a E[X]$ $\qquad (b = 0)$
- $E[b] = b$ $\qquad (a = 0$

Defn
The n^{th} moment of X $(n = 0, 1, 2, \ldots)$ is

$$E[X^n]$$

Thus the 1^{st} moment of X is the mean $E[X]$.

<u>Defn</u>

The <u>Variance</u> of X is the mean of the r.v. $(X-E[X])^2$, i.e.

$$Var(X) = E\left[(X-E[X])^2\right]$$

note since $(X-E[X])^2 \geq 0$, we have $Var(X) \geq 0$.

Variance is a measure of how "spread out" the "mass" of X is.

<u>Defn</u>
The <u>standard deviation</u> of X is

$$\sigma_X = \sqrt{Var(X)}$$

Ex. (Previous)

$$p_X(x) = \begin{cases} .2 & \text{if } x = -1 \\ .3 & x = 1 \\ .5 & x = 2 \end{cases}$$

We computed

$$E[X] = 1.1 \qquad (1^{st} \text{ moment})$$

$$E[X^2] = 2.5 \qquad (2^{nd} \text{ moment})$$

Now

$$Var(X) = E\left[(X - E[X])^2\right]$$

$$= \sum_x (x - (1.1))^2 \, p_X(x)$$

$$= (-1-(1.1))^2 \cdot (.2) + (1-(1.1))^2 (.3) + (2-(1.1))^2 (.5)$$

$$= \boxed{1.29}$$

and $\sigma_X = \boxed{1.1358}$

Simpler formula:

Variance in terms of Moments

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2$$

Proof

$$\mathrm{Var}(X) = E[(X - \overbrace{E[X]}^{\mu})^2]$$

$$= \sum_x (x-\mu)^2 p_X(x)$$

$$= \sum_x (x^2 - 2\mu x + \mu^2) p_X(x)$$

$$= \sum_x x^2 p_X(x) - 2\mu \sum_x x p_X(x) + \mu^2 \cdot \sum_x p_X(x)$$

$$= E[X^2] - 2(E[X])^2 + (E[X])^2$$

$$= E[X^2] - (E[X])^2 \qquad \blacksquare$$

Ex. (previous)

we had $E[X] = 1.1$, $E[X^2] = 2.5$

Thus

$$Var(X) = (2.5) - (1.1)^2 = \boxed{1,29}$$

Ex. recall if $Y = aX + b$ then

$$E[Y] = E[aX+b] = aE[X] + b$$

The 2^{nd} moment of Y is

$$E[Y^2] = E[(aX+b)^2]$$

$$= \sum_x (ax+b)^2 p_X(x)$$

$$= \sum_x (a^2x^2 + 2abx + b^2) p_X(x)$$

$$= a^2 \sum_x x^2 p_X(x) + 2ab \sum_x x p_X(x) + b^2 \sum_x p_X(x)$$

$$= a^2 E[X^2] + 2ab\, E[X] + b^2$$

Now Variance of Y:

$$Var(Y) = Var(aX + b)$$

$$= E[(aX+b)^2] - E[aX+b]^2$$

$$= a^2 E[X^2] + 2ab\, E[X] + b^2 - (aE[X] + b)^2$$

$$= a^2 E[X^2] + \cancel{2ab\, E[X] + b^2}$$

$$- (a^2 E[X]^2 + \cancel{2ab\, E[X] + b^2})$$

$$= a^2 (E[X^2] - E[X]^2)$$

$$= a^2 \cdot Var(X)$$

Summary:

- $E[aX+b] = a\,E[X]+b$
- $Var(aX+b) = a^2\,Var(X)$

also

- $\sigma_{aX+b} = |a|\,\sigma_X$

Mean and Variance of Common r.v.s

Bernoulli r.v.

$$P_X(k) = \begin{cases} p & \text{if } k=1 \\ 1-p & k=0 \end{cases}$$

$$E[X] = 1 \cdot p + 0 \cdot (1-p) = p$$

$$E[X^2] = 1^2 \cdot p + 0^2 \cdot (1-p) = p$$

$$Var(X) = E[X^2] - E[X]^2$$

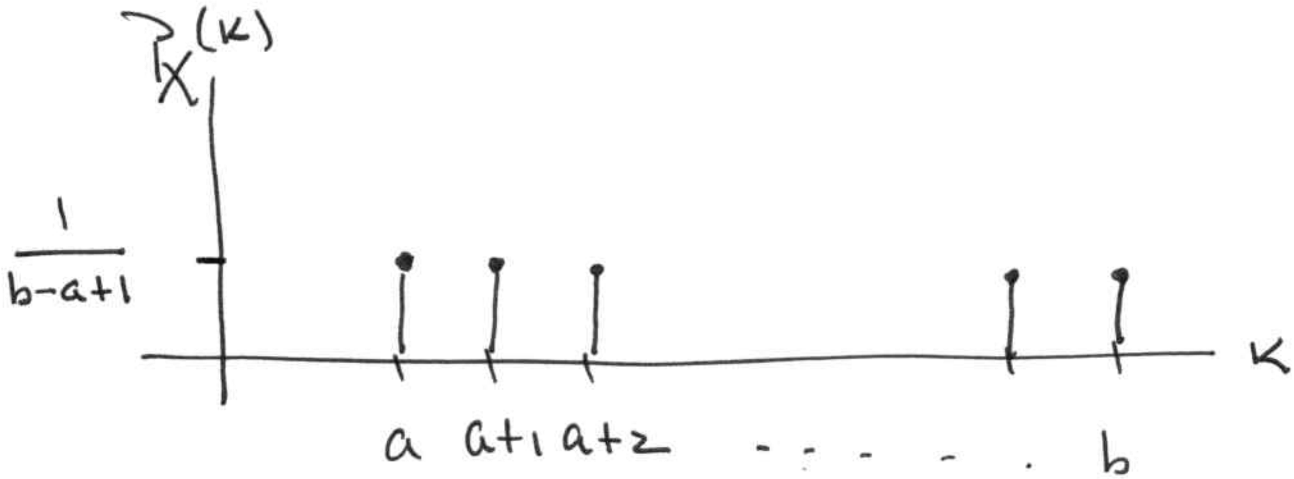
$$= p - p^2$$

$$= p(1-p)$$

Discrete Uniform r.v.

Let $a, b \in \mathbb{Z}$, and suppose X distributes its 'mass' uniformly between a and b, i.e. same probability on each int k in range

$$a \le k \le b$$



so

$$P_X(k) = \begin{cases} \frac{1}{b-a+1} & \text{if } k = a, a+1, \dots, b \\ 0 & \text{otherwise} \end{cases}$$

so

$$E[X] = \sum_{k=a}^{b} k \cdot \frac{1}{b-a+1}$$

$$= \left(\frac{1}{b-a+1}\right) \cdot \sum_{k=a}^{b} k$$

$$= \left(\frac{1}{b-a+1}\right)\left(\sum_{k=1}^{b} k - \sum_{k=1}^{a-1} k\right)$$

$$= \frac{1}{b-a+1}\left(\frac{b(b+1)}{2} - \frac{a(a-1)}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{b-a+1}\right)\left((b^2-a^2)+(b+a)\right)$$

$$= \frac{1}{2}\left(\frac{1}{b-a+1}\right)\left((b-a)(b+a)+(b+a)\right)$$

$$= \frac{1}{2}\left(\frac{1}{\cancel{b-a+1}}\right)(a+b)(\cancel{b-a+1})$$

$$= \frac{a+b}{2}$$

To compute $Var(X)$, first set

$$Y = X + (1-a)$$

and note:

$$Var(Y) = Var(X+(1-a)) = Var(X).$$

let $n = b-a+1$, and observe Y is uniform over $1 \le k \le n$, since

$$X = a \Rightarrow Y = a + (1-a) = 1$$
$$X = b \Rightarrow Y = b+1-a = b-a+1 = n$$

Thus $E[Y] = \frac{n+1}{2}$, and

$$E[Y^2] = \sum_{k=1}^{n} k^2 \cdot \frac{1}{n}$$

$$= \frac{1}{n} \sum_{k=1}^{n} k^2$$

$$= \frac{1}{\cancel{n}} \cdot \frac{\cancel{n}(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(2n+1)}{6}$$

Therefore

$$\mathrm{Var}(Y) = \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$$

$$\vdots$$

$$= \frac{(n+1)(n-1)}{12}$$

$$= \frac{(b-a)(b-a+2)}{12}.$$