CSE 107 10-10-24

Recall: <u>Geometric random variable</u>

$X(\omega) = \#$ trials until 1^{st} success in a (unlimited) seq. of independent Bernoulli trials.

$$\text{range}(X) = \{1, 2, 3, \ldots\} = \mathbb{Z}^+$$

Thus

$$P_X(k) = P(X = k)$$

$$= (1-p)^{k-1} \cdot p \quad \text{for } k = 1, 2, \ldots$$

note

$$\sum_{K=1}^{\infty} P_X(K) = \sum_{K=1}^{\infty} p(1-p)^{K-1}$$

$$= p \cdot \sum_{K=1}^{\infty} (1-p)^{K-1}$$

$$= p \cdot \sum_{K=0}^{\infty} (1-p)^{K}$$

$$= p \cdot \frac{1}{1-(1-p)}$$

$$= p \cdot \frac{1}{p} = \boxed{1}$$

Ex. What is $P(X \geq 5)$

$\uparrow$ geom.

$$P(X \geq 5) = \sum_{k=5}^{\infty} p_X(k)$$

$$= \sum_{k=5}^{\infty} p \cdot (1-p)^{k-1}$$

$$= p \cdot \sum_{k=0}^{\infty} (1-p)^{k+4} \quad \left\{ \begin{array}{l} \text{replace} \\ k \leftarrow k+5 \end{array} \right.$$

$$= p(1-p)^4 \cdot \sum_{k=0}^{\infty} (1-p)^k$$

$$= p \cdot (1-p)^4 \cdot \frac{1}{1-(1-p)}$$

$$= \boxed{(1-p)^4}.$$

The Poisson Random Variable

A Poisson r.v. has PMF

$$P_X(k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!} \quad (\text{for } k = 0, 1, 2, \ldots)$$

Parameter: $\lambda > 0$

First observe

$$\sum_{k=0}^{\infty} p_X(k) = \sum_{k=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \cdot e^{\lambda} = 1 \quad \checkmark$$

we have the Binomial-Poisson approximation

$$e^{-\lambda} \cdot \frac{\lambda^k}{k!} \approx \binom{n}{k} p^k (1-p)^{n-k} \quad \left\{ \begin{array}{l} \text{for} \\ k=0,..,n \end{array} \right.$$

when $\lambda = n \cdot p$, n is large and p small

more precisely:
let $\lambda > 0$ be constant, and $p = \frac{\lambda}{n}$.

Then

$$\lim_{n \to \infty} \binom{n}{k} \cdot \left(\frac{\lambda}{n}\right)^k \cdot \left(1 - \frac{\lambda}{n}\right)^{n-k} = e^{-\lambda} \cdot \frac{\lambda^k}{k!}.$$

Ex.

rate

Let $n = 200$, $\lambda = 2$, $p = \frac{\lambda}{n} = .01$

The prob. of $k = 4$ successes in 200 trials with prob. .01 of success is

$$\binom{200}{4}(.01)^4(.99)^{196} = .0902197\ldots$$

Poisson approx.

$$e^{-2} \cdot \frac{2^4}{4!} = .0902235\ldots$$

2.3 Functions of a rand. var

Given a r.v. $X : \Omega \to \mathbb{R}$,

we can compose it with a fcn.

$$g : \mathbb{R} \to \mathbb{R}$$

i.e.

$$\begin{tikzcd} \Omega \arrow[r, "X"] \arrow[rr, bend right, "g \circ X = g(X)"'] & \mathbb{R} \arrow[r, "g"] & \mathbb{R} \end{tikzcd}$$

Let $Y = g(X)$.

Ex Suppose X has PMF

$$P_X(x) = \begin{cases} .2 & \text{if } x = -1 \\ .3 & x = 1 \\ .5 & x = 2 \\ 0 & \text{otherwise} \end{cases}$$

Let $Y = X^2$. Find the PMF of Y.

$$P_Y(y) = P(Y = y)$$

$$= P(X^2 = y)$$

$$= P(X = \pm\sqrt{y})$$

$$= P(X = \sqrt{y} \text{ or } X = -\sqrt{y})$$

$$= P(X = \sqrt{y}) + P(X = -\sqrt{y})$$

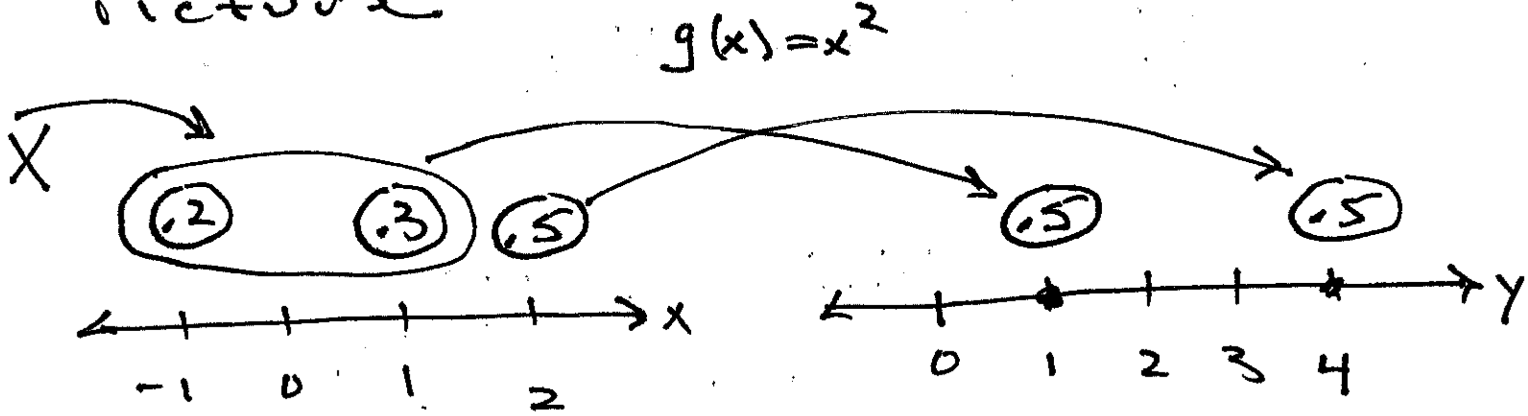
$$= p_X(\sqrt{y}) + p_X(-\sqrt{y})$$

$$= \begin{cases} \cancel{.2 \quad \sqrt{y} = -1} \\ .3 \quad \sqrt{y} = 1 \\ .5 \quad \sqrt{y} = 2 \end{cases} + \begin{cases} .2 \quad -\sqrt{y} = -1 \\ \cancel{.3 \quad -\sqrt{y} = 1} \\ \cancel{.5 \quad -\sqrt{y} = 2} \end{cases}$$

$$= \begin{cases} .3 & y = 1 \\ .5 & y = 4 \end{cases} + \begin{cases} .2 & y = 1 \\ - & - - - \end{cases}$$

$$= \begin{cases} .5 & y = 1 \\ .5 & y = 4 \\ 0 & \text{otherwise} \end{cases}$$

Picture



2.4 Expectation, mean & Variance

Defn

The expected value of a r.v. X is

$$E[X] = \sum_{x} x\, P_X(x) \quad \left\{ \begin{array}{l} \text{where sum} \\ \text{over all} \\ x \in \text{range}(X) \end{array} \right.$$

Also called the mean, or average of X. This is the 'center of mass' of probability.

Ex. (Previous)

$$p_X(x) = \begin{cases} .2 & \text{if } x=-1 \\ .3 & x=1 \\ .5 & x=2 \end{cases}$$

$$\therefore E[X] = (-1)(.2) + (1)\cdot(.3) + (2)(.5)$$

$$= \boxed{1.1}$$

also for $Y = X^2$

$$p_Y(y) = \begin{cases} .5 & y=1 \\ .5 & y=4 \end{cases}$$

$$\therefore E[X^2] = E[Y] = 1\cdot(.5) + 4\cdot(.5)$$

$$= \boxed{2.5}$$

note: $E[X^2] \neq E[X]^2$

Expected value rule

let $Y = g(X)$. Then

$$E[Y] = E[g(X)] = \sum_x g(x) \cdot p_X(x)$$

Ex. (Previous)

$$E[X^2] = (-1)^2(.2) + (1)^2 \cdot (.3) + (2)^2(.5)$$

$$= \dots = \boxed{2.5}$$