CSE 107 10-1-24

Recall:

$\{\max = 4\}$

Ω:

11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66

$\{1^{st} > 2^{nd}\}$

$P(\max = 4) = \frac{7}{36}$

$P(1^{st}\text{ roll} > 2^{nd}\text{ roll}) = \frac{15}{36}$

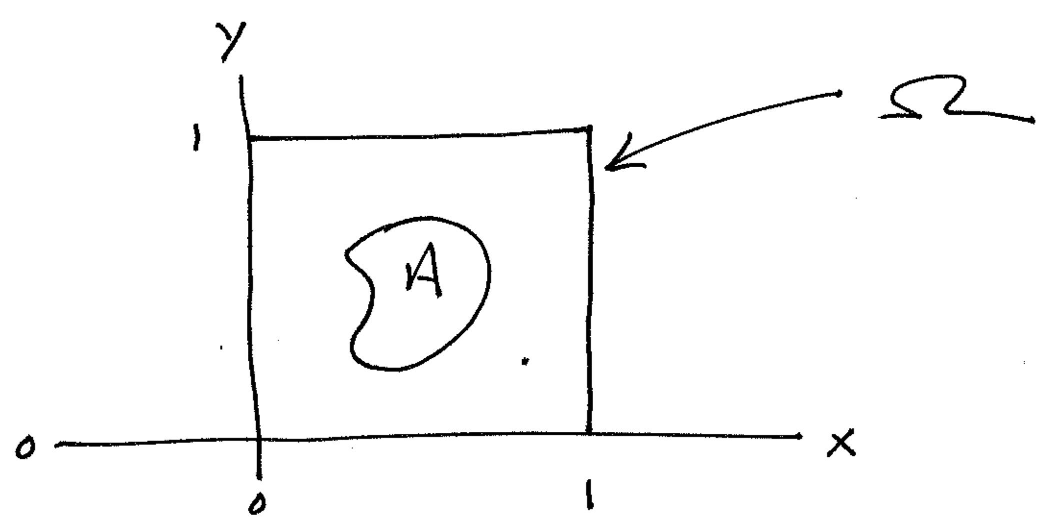
Continuous Models

If Ω is a 'continuum',

i.e. an interval, region in $\mathbb{R}\times\mathbb{R}$

or in $\mathbb{R}^3$, then $P(\omega)$

for each $\omega \in \Omega$, is not sufficient.



In this case Uniform Probability law means

$$P(A) = c \cdot \text{area}(A)$$

what is c?

$$1 = P(\Omega) = c \cdot \text{area}(\Omega)$$

$$\therefore \quad c = \frac{1}{\text{area}(\Omega)}$$

$$\therefore \quad \boxed{P(A) = \frac{\text{area}(A)}{\text{area}(\Omega)}}$$

If $\text{area}(\Omega) = 1$, then

$$P(A) = \text{area}(A).$$

Ex.

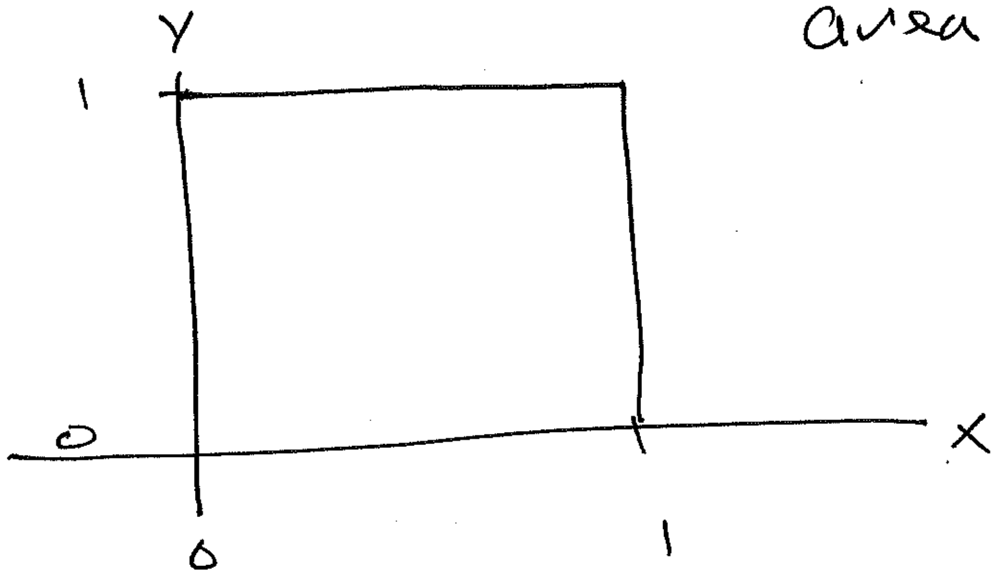
R. & J. have a date and each will be delayed by a 'random' time in $[0, 1]$ hour

1^{st} to arrive will wait 15 minutes, then leave.

Let x, y be R. & J.'s delay times: $0 \le x \le 1$ and $0 \le y \le 1$

Assume all delay pairs (x, y) are 'equally likely'

Ω is



$\text{area}(\Omega) = 1$

find $P(M)$ where

$M = \{R \text{ and } J \text{ meet}\}$

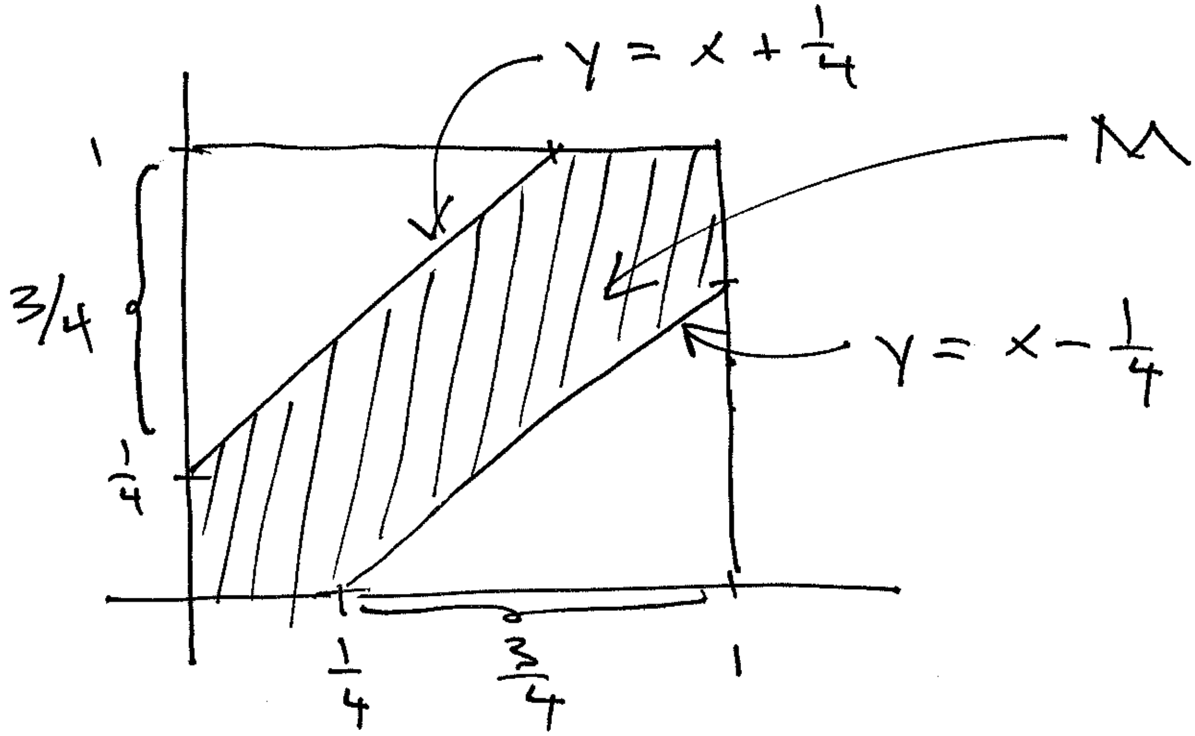
$= \{(x,y) \mid |x-y| \le \frac{1}{4},\ (x,y) \in \Omega\}$

<u>note</u>: $|x-y| \le \frac{1}{4}$

iff $-\frac{1}{4} \le x-y \le \frac{1}{4}$

iff $-\frac{1}{4} \le x-y$ and $x-y \le \frac{1}{4}$

iff $y \leq x + \frac{1}{4}$ and $y \geq x - \frac{1}{4}$



$$P(M) = \text{area}(M)$$

$$= 1 - \left(\frac{3}{4}\right)^2 = 1 - \frac{9}{16}$$

$$= \boxed{\frac{7}{16}}$$

Exercise

Alice & Bob each pick a random # in $[0,2]$, call Alice's x, and Bob's y.

$$\Omega = [0,2] \times [0,2]$$

find Probability that both

$$2x > y^2 \text{ and } 2y > x^2$$

Properties of $P(\cdot)$

(a) if $A \subseteq B$, then $P(A) \leq P(B)$

(b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

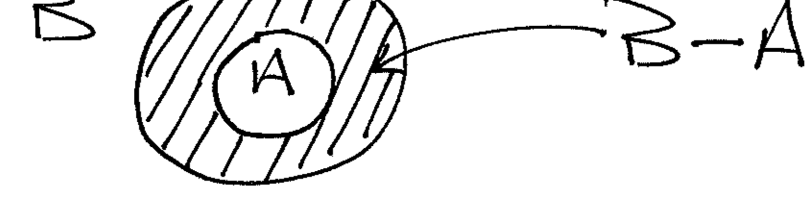
(c) $P(A \cup B) \leq P(A) + P(B)$

(d) $P(A \cup B \cup C)$

$$= P(A) + P(A^c \cap B) + P(A^c \cap B^c \cap C)$$

(a) Proof

$$A \subseteq B \Rightarrow \begin{cases} B = A \cup (B - A) \\ A \cap (B - A) = \phi \end{cases}$$

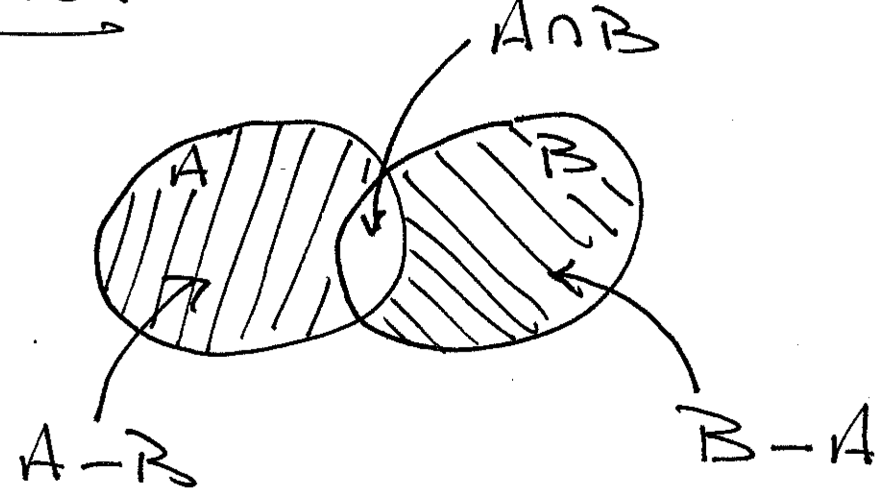


Thus

$$P(B) = P(A) + P(B-A) \geq P(A)$$

$$\therefore \quad P(A) \leq P(B) \qquad \blacksquare$$

(b) <u>Proof</u>



<u>note</u>: $A \cap B$, $A-B$, $B-A$ are pairwise disjoint, and

$$A \cup B = (A-B) \cup (A \cap B) \cup (B-A)$$

so

$$P(A \cup B) = P(A-B) + P(A \cap B) + P(B-A)$$

<u>note</u>:

$$B = (A \cap B) \cup (B - A)$$

$$A = (A \cap B) \cup (A - B)$$

$$\therefore \quad P(B) = P(A \cap B) + P(B - A)$$

$$\therefore \quad P(B-A) = P(B) - P(A \cap B)$$

similarly: $\quad P(A-B) = P(A) - P(A \cap B)$

Thus

$$P(A \cup B) = \left[P(A) - \cancel{P(A \cap B)}\right] + \cancel{P(A \cap B)}$$

$$+ \left[P(B) - P(A \cap B)\right]$$

and

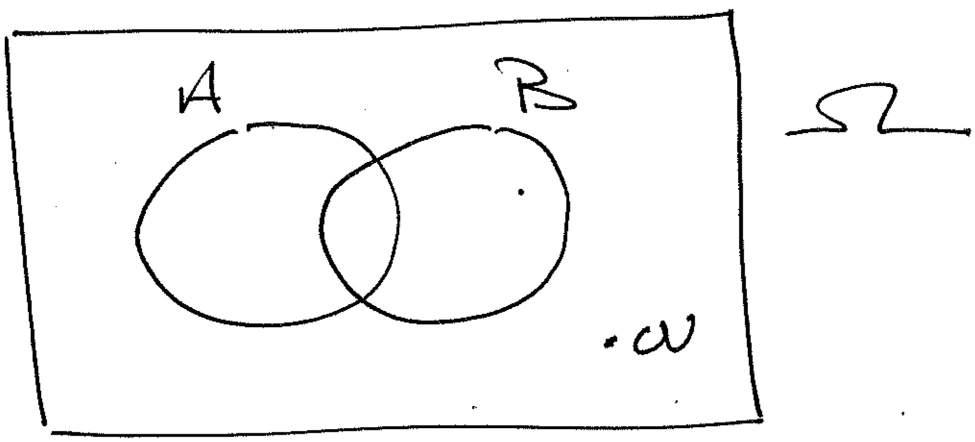
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

(c) follows from (b).

<u>Proof of (d)</u>: Exercise.

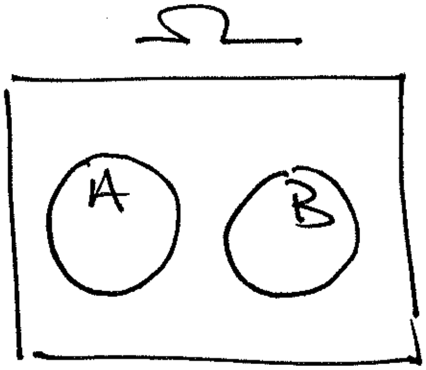
(1.3) Conditional Probability

suppose $A, B \subseteq \Omega$



In general, the information that B occurs affects the probability that A occurs.

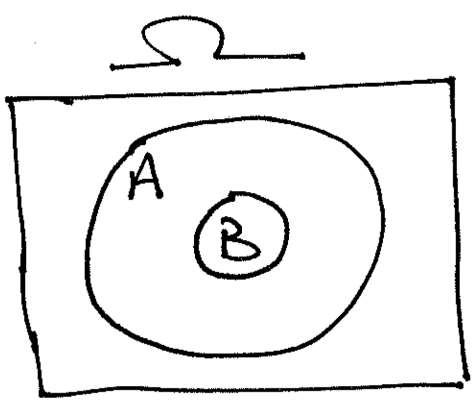
Two extreme cases:

• 

$A \cap B = \phi$

$\therefore A \subseteq B^c, \; B \subseteq A^c$

$\therefore$ B occurs $\Rightarrow$ A <u>does not</u> occur.

• 

$B \subseteq A$

$\therefore$ B occurs $\Rightarrow$ A <u>does</u> occur

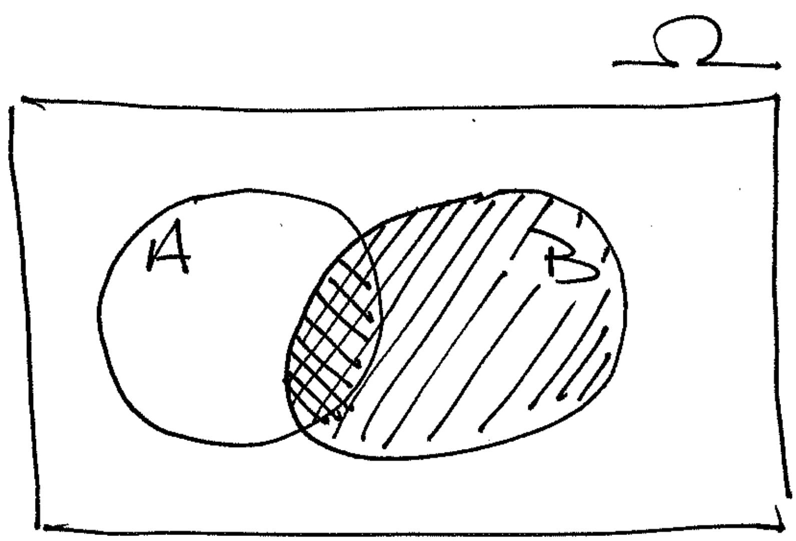
Defn

The conditional Probability of A given B, denoted $P(A|B)$, is

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

(where we assume $P(B) > 0$).

Picture



Notice:

$P(\cdot \mid B)$ is a valid Probability law, in that axioms are satisfied.

(i) $P(A|B) = \dfrac{P(A \cap B)}{P(B)} \geq 0$ ✓

(ii) $P(\Omega|B) = \dfrac{P(\Omega \cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$ ✓

(iii) $P(A_1 \cup A_2|B) = \dfrac{P((A_1 \cup A_2) \cap B)}{P(B)}$ $\left\{ A_1 \cap A_2 = \phi \right.$

$\Downarrow$

$= \dfrac{P((A_1 \cap B) \cup (A_2 \cap B))}{P(B)}$ $\left\{ (A_1 \cap B) \cap (A_2 \cap B) = \phi \right.$

$$= \frac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)}$$

$$= \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)}$$

$$= P(A_1 | B) + P(A_2 | B) \quad \checkmark$$

so all theorems that hold for $P(\cdot)$ also hold for $P(\cdot | B)$

For instance

- $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

Becomes

- $P(A_1 \cup A_2 | B) = P(A_1|B) + P(A_2|B) - P(A_1 \cap A_2 | B)$

<u>note</u>:

if Ω is finite and $P(\cdot)$ is discrete uniform, then

$$P(A|B) = \frac{\frac{|A \cap B|}{\cancel{|\Omega|}}}{\frac{|B|}{\cancel{|\Omega|}}} = \frac{|A \cap B|}{|B|}$$

note

defn: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

given

$$P(A \cap B) = P(A|B) \cdot P(B)$$