**CSE 107**
**Probability and Statistics for Engineers**
**Information for Final Exam**

**Expectation, Variance and Covariance Formulas:**

$E[aX + b] = aE[X] + b$

$E[X + Y] = E[X] + E[Y]$

$Var(aX + b) = a^2 Var(X)$

$Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$

Iterated Expectation Law: $E\big[E[X|Y]\big] = E[X]$

Total Variance Law: $Var(X) = E[Var(X|Y)] + Var(E[X|Y])$

**Random Variables:**

<u>Discrete Uniform on $[a, b] = \{a, a + 1, a + 2, \ldots, b\}$, where $a, b \in \mathbb{Z}$</u>

$$p_X(k) = \begin{cases} \dfrac{1}{b - a + 1} & \text{if } a \le k \le b \\ 0 & \text{otherwise} \end{cases} \qquad F_X(x) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{\lfloor x \rfloor - a + 1}{b - a + 1} & a \le x < b \\ 1 & x \ge b \end{cases}$$

$$E[X] = \frac{a + b}{2} \qquad Var(X) = \frac{(b - a)(b - a + 2)}{12}$$

<u>Bernoulli with parameter $p$</u>

$$p_X(k) = \begin{cases} p & \text{if } k = 1 \\ 1 - p & \text{if } k = 0 \end{cases} \qquad F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - p & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

$$E[X] = p \qquad Var(X) = p(1 - p)$$

<u>Binomial with parameters $n, p$</u>

$$p_X(k) = \begin{cases} \binom{n}{k} p^k (1 - p)^{n-k} & \text{if } 0 \le k \le n \\ 0 & \text{otherwise} \end{cases} \qquad F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \displaystyle\sum_{k=0}^{\lfloor x \rfloor} \binom{n}{k} p^k (1 - p)^{n-k} & 0 \le x < n \\ 1 & x \ge n \end{cases}$$

$$E[X] = np \qquad Var(X) = np(1 - p)$$

Geometric with parameter $p$

$$p_X(k) = \begin{cases} (1-p)^{k-1} \cdot p & \text{if } k = 1, 2, \dots \\ 0 & \text{otherwise} \end{cases} \qquad F_X(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 - (1-p)^{\lfloor x \rfloor} & x \geq 1 \end{cases}$$

$$E[X] = \frac{1}{p} \qquad Var(X) = \frac{1-p}{p^2}$$

Poisson with parameter $\lambda$

$$p_X(k) = \begin{cases} e^{-\lambda} \cdot \frac{\lambda^k}{k!} & \text{if } k = 0, 1, \dots \\ 0 & \text{otherwise} \end{cases} \qquad F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \sum_{k=0}^{\lfloor x \rfloor} e^{-\lambda} \cdot \frac{\lambda^k}{k!} & x \geq 0 \end{cases}$$

$$E[X] = \lambda \qquad Var(X) = \lambda$$

Continuous Uniform on $[a, b]$ where $a, b \in \mathbb{R}$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases} \qquad F_X(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & a \leq x < b \\ 1 & x \geq b \end{cases}$$

$$E[X] = \frac{a+b}{2} \qquad Var(X) = \frac{(b-a)^2}{12}$$

Exponential with parameter $\lambda$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$

$$E[X] = \frac{1}{\lambda} \qquad Var(X) = \frac{1}{\lambda^2}$$

Normal with mean $\mu$ and variance $\sigma^2$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad F_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}}\, dt$$

$$E[X] = \mu \qquad Var(X) = \sigma^2$$

Standard Normal ($\mu = 0$ and $\sigma = 1$)

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt$$

$$E[X] = 0 \qquad Var(X) = 1$$

**Distributions Associated with Random Processes**

Pascal Distribution of order $k$

Let $Y_k$ be the $k^{\text{th}}$ arrival time in a Bernoulli process with parameter $p$. Then $Y_k = T_1 + T_2 + \cdots + T_k$ where $T_i$ are independent geometric random variables with paramenter $p$.

$$p_{Y_k}(t) = \binom{t-1}{k-1} p^k (1-p)^{t-k} \qquad \text{for } t = k, k+1, k+2, \ldots$$

$$E[Y_k] = \frac{k}{p}$$

$$Var(Y_k) = \frac{k(1-p)}{p^2}$$

Erlang Distribution of order $k$

Let $Y_k$ be the $k^{\text{th}}$ arrival time in a Poisson process with parameter $\lambda$. Then $Y_k = T_1 + T_2 + \cdots + T_k$ where $T_i$ are independent exponential random variables with paramenter $\lambda$.

$$f_{Y_k}(t) = \frac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!} \qquad \text{for } t \in [0, \infty)$$

$$E[Y_k] = \frac{k}{\lambda}$$

$$Var(Y_k) = \frac{k}{\lambda^2}$$

**Chapman-Kolmogorov Equations**

Consider a Markov chain model with state space $S = \{1, 2, \ldots, m\}$ and state transition probabilities $p_{ij}$ for $i, j \in S$. Let $X_n$ denote the state at time $n \geq 0$, and let $r_{ij}(n) = P(X_n = j \mid X_0 = i)$ be the $n$-step transition probabilities. Then $r_{ij}(n)$ satisfies the following recurrence.

$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1) \cdot p_{kj} \qquad \text{for all } i \in S \text{ and } n \geq 1$$

# Standard Normal Cumulative Distribution Function

| | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| **0.0** | .5000 | .5040 | .5080 | .5120 | .5160 | .5199 | .5239 | .5279 | .5319 | .5359 |
| **0.1** | .5398 | .5438 | .5478 | .5517 | .5557 | .5596 | .5636 | .5675 | .5714 | .5753 |
| **0.2** | .5793 | .5832 | .5871 | .5910 | .5948 | .5987 | .6026 | .6064 | .6103 | .6141 |
| **0.3** | .6179 | .6217 | .6255 | .6293 | .6331 | .6368 | .6406 | .6443 | .6480 | .6517 |
| **0.4** | .6554 | .6591 | .6628 | .6664 | .6700 | .6736 | .6772 | .6808 | .6844 | .6879 |
| **0.5** | .6915 | .6950 | .6985 | .7019 | .7054 | .7088 | .7123 | .7157 | .7190 | .7224 |
| **0.6** | .7257 | .7291 | .7324 | .7357 | .7389 | .7422 | .7454 | .7486 | .7517 | .7549 |
| **0.7** | .7580 | .7611 | .7642 | .7673 | .7704 | .7734 | .7764 | .7794 | .7823 | .7852 |
| **0.8** | .7881 | .7910 | .7939 | .7967 | .7995 | .8023 | .8051 | .8078 | .8106 | .8133 |
| **0.9** | .8159 | .8186 | .8212 | .8238 | .8264 | .8289 | .8315 | .8340 | .8365 | .8389 |
| **1.0** | .8413 | .8438 | .8461 | .8485 | .8508 | .8531 | .8554 | .8577 | .8599 | .8621 |
| **1.1** | .8643 | .8665 | .8686 | .8708 | .8729 | .8749 | .8770 | .8790 | .8810 | .8830 |
| **1.2** | .8849 | .8869 | .8888 | .8907 | .8925 | .8944 | .8962 | .8980 | .8997 | .9015 |
| **1.3** | .9032 | .9049 | .9066 | .9082 | .9099 | .9115 | .9131 | .9147 | .9162 | .9177 |
| **1.4** | .9192 | .9207 | .9222 | .9236 | .9251 | .9265 | .9279 | .9292 | .9306 | .9319 |
| **1.5** | .9332 | .9345 | .9357 | .9370 | .9382 | .9394 | .9406 | .9418 | .9429 | .9441 |
| **1.6** | .9452 | .9463 | .9474 | .9484 | .9495 | .9505 | .9515 | .9525 | .9535 | .9545 |
| **1.7** | .9554 | .9564 | .9573 | .9582 | .9591 | .9599 | .9608 | .9616 | .9625 | .9633 |
| **1.8** | .9641 | .9649 | .9656 | .9664 | .9671 | .9678 | .9686 | .9693 | .9699 | .9706 |
| **1.9** | .9713 | .9719 | .9726 | .9732 | .9738 | .9744 | .9750 | .9756 | .9761 | .9767 |
| **2.0** | .9772 | .9778 | .9783 | .9788 | .9793 | .9798 | .9803 | .9808 | .9812 | .9817 |
| **2.1** | .9821 | .9826 | .9830 | .9834 | .9838 | .9842 | .9846 | .9850 | .9854 | .9857 |
| **2.2** | .9861 | .9864 | .9868 | .9871 | .9875 | .9878 | .9881 | .9884 | .9887 | .9890 |
| **2.3** | .9893 | .9896 | .9898 | .9901 | .9904 | .9906 | .9909 | .9911 | .9913 | .9916 |
| **2.4** | .9918 | .9920 | .9922 | .9925 | .9927 | .9929 | .9931 | .9932 | .9934 | .9936 |
| **2.5** | .9938 | .9940 | .9941 | .9943 | .9945 | .9946 | .9948 | .9949 | .9951 | .9952 |
| **2.6** | .9953 | .9955 | .9956 | .9957 | .9959 | .9960 | .9961 | .9962 | .9963 | .9964 |
| **2.7** | .9965 | .9966 | .9967 | .9968 | .9969 | .9970 | .9971 | .9972 | .9973 | .9974 |
| **2.8** | .9974 | .9975 | .9976 | .9977 | .9977 | .9978 | .9979 | .9979 | .9980 | .9981 |
| **2.9** | .9981 | .9982 | .9982 | .9983 | .9984 | .9984 | .9985 | .9985 | .9986 | .9986 |
| **3.0** | .9987 | .9987 | .9987 | .9988 | .9988 | .9989 | .9989 | .9989 | .9990 | .9990 |
| **3.1** | .9990 | .9991 | .9991 | .9991 | .9992 | .9992 | .9992 | .9992 | .9993 | .9993 |
| **3.2** | .9993 | .9993 | .9994 | .9994 | .9994 | .9994 | .9994 | .9995 | .9995 | .9995 |
| **3.3** | .9995 | .9995 | .9995 | .9996 | .9996 | .9996 | .9996 | .9996 | .9996 | .9997 |
| **3.4** | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9998 |