CSE 107
Probability and Statistics for Engineers
The Gaussian Integral

The following integrals are used to establish some basic facts about the Normal distribution. The first is often just called the *Gaussian Integral*.

(1) $$\int_{-\infty}^{\infty} e^{-x^2}\, dx \;=\; \sqrt{\pi}$$

Proof:
Square the left hand side, then switch the double integral to polar coordinates.

$$\left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)\cdot\left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\; dxdy$$

$$= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2}\; rdrd\theta$$

$$= \int_{0}^{2\pi} d\theta \cdot \int_{0}^{\infty} re^{-r^2}\, dr$$

$$= 2\pi\cdot\left(-\frac{1}{2}\right)e^{-r^2}\Big|_{0}^{\infty}$$

$$= \;-\pi\cdot(0-1)$$

$$= \pi$$

proving (1). ■

(2) $$\int_{-\infty}^{\infty} e^{-u^2/2}\; du = \;\sqrt{2\pi}$$

Proof:
Do the substitution $x = u/\sqrt{2}$, to get $x^2 = u^2/2$ and $du = \sqrt{2}\, dx$. Using (1) we have

$$\int_{-\infty}^{\infty} e^{-u^2/2}\; du = \int_{-\infty}^{\infty} e^{-x^2}\sqrt{2}dx = \;\sqrt{2}\cdot\sqrt{\pi} = \sqrt{2\pi}$$

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The following identity establishes that the Normal density is a valid PDF.

(3) $$\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = 1$$

Proof:
Do the substitution $u = (x-\mu)/\sigma \implies \sigma du = dx$ and apply (2) to get

$$\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} e^{-u^2/2}\,\sigma\,du = \frac{1}{\sqrt{2\pi}}\cdot\sqrt{2\pi} = 1$$

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We can use the same substitution to calculate the variance of the Normal distribution X.

(4) $Var(X) = \sigma^2$

Proof:
Here we use a different variable to do the substitution: $y = (x-\mu)/\sigma \implies dx = \sigma\,dy$

$$\begin{aligned} Var(X) &= E[(X-\mu)^2] \\ &= \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty}(x-\mu)^2\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx \\ &= \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty}\sigma^2 y^2\, e^{-y^2/2}\,\sigma\,dy \\ &= \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y^2\, e^{-\frac{y^2}{2}}\,dy \end{aligned}$$

Now integration by parts with $u = y, dv = ye^{y^2/2} \implies du = dy, v = -e^{-y^2/2}$, and (2) yields

$$\begin{aligned} Var(X) &= \frac{\sigma^2}{\sqrt{2\pi}}\left(-ye^{-\frac{y^2}{2}}\right)\Big|_{-\infty}^{\infty} - \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} -e^{-\frac{y^2}{2}}\,dy \\ &= 0 + \frac{\sigma^2}{\sqrt{2\pi}}\cdot\sqrt{2\pi} \\ &= \sigma^2 \end{aligned}$$

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Many other import facts about the Gaussian integral will be covered in lecture. See here for a proof that the sum of two Normal random variables is again normal, which uses the convolution product and a lot of algebraic details.