CSE 107
Probability and Statistics for Engineers
Approximation of the Binomial PMF by the Poisson PMF

Suppose that $n \geq 1$ is large, $p \in (0, 1)$ is small and

$$\lambda = np.$$

We prove that under these conditions, the Poisson PMF with parameter λ is a good approximation to the Binomial PMF with parameters n and p. In other words, for any k in the range $0 \leq k \leq n$, we have

$$e^{-\lambda} \cdot \frac{\lambda^k}{k!} \approx \binom{n}{k} p^k (1-p)^{n-k}.$$

More precisely, we set $p = \lambda/n$ and prove the limit

(1) $$\lim_{n\to\infty} \left[\binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \left(\frac{\lambda}{n}\right)\right)^{n-k} \right] = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

We begin by establishing the following limits, which hold for any $\lambda > 0$ and $k \geq 0$:

(2) $$\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}$$

and

(3) $$\lim_{n\to\infty} \left(\frac{n!}{(n-k)!\,(n-\lambda)^k} \right) = 1.$$

Proof of (2):
Take the natural log of the sequence, and observe

$$\ln\left(1 - \frac{\lambda}{n}\right)^n = n \cdot \ln\left(1 - \frac{\lambda}{n}\right) = \frac{\ln\left(1 - \frac{\lambda}{n}\right)}{\frac{1}{n}}.$$

By l'Hopital's rule we obtain

$$\lim_{n\to\infty} \left[\ln\left(1 - \frac{\lambda}{n}\right)^n \right] = \lim_{n\to\infty} \left[\frac{\frac{1}{(1-\lambda/n)} \cdot \frac{\lambda}{n^2}}{-\frac{1}{n^2}} \right] = \lim_{n\to\infty} \left(\frac{-\lambda}{1-\lambda/n} \right) = -\lambda.$$

Since $e^x = \exp(x)$ is a continuous function, it follows that

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = \lim_{n\to\infty}\exp\left[\ln\left(1-\frac{\lambda}{n}\right)^n\right]$$

$$= \exp\left(\lim_{n\to\infty}\left[\ln\left(1-\frac{\lambda}{n}\right)^n\right]\right)$$

$$= \exp(-\lambda) = e^{-\lambda},$$

establishing the claim. ■

Proof of (3):
By canceling factors top and bottom we write

$$\frac{n!}{(n-k)!\,(n-\lambda)^k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{(n-\lambda)^k}$$

$$= \left(\frac{n}{n-\lambda}\right)\left(\frac{n-1}{n-\lambda}\right)\left(\frac{n-2}{n-\lambda}\right)\cdots\left(\frac{n-k+1}{n-\lambda}\right).$$

For each i in the range $0 \le i \le k-1$,

$$\lim_{n\to\infty}\left(\frac{n-i}{n-\lambda}\right) = \lim_{n\to\infty}\left(\frac{1-\frac{i}{n}}{1-\frac{\lambda}{n}}\right) = 1,$$

and therefore

$$\lim_{n\to\infty}\left(\frac{n!}{(n-k)!\,(n-\lambda)^k}\right) = 1\cdot 1\cdot 1\cdots 1 = 1,$$

as required. ■

Proof of (1):
We have

$$\binom{n}{k}\cdot\left(\frac{\lambda}{n}\right)^k\cdot\left(1-\frac{\lambda}{n}\right)^{n-k} = \frac{n!}{k!\,(n-k)!}\cdot\left(\frac{\lambda}{n}\right)^k\cdot\frac{\left(1-(\lambda/n)\right)^n}{\left(1-(\lambda/n)\right)^k}$$

$$= \frac{n!}{k!\,(n-k)!}\cdot\left(\frac{\lambda/n}{1-(\lambda/n)}\right)^k\cdot\left(1-\frac{\lambda}{n}\right)^n$$

$$= \frac{n!}{k!\,(n-k)!}\cdot\left(\frac{\lambda}{n-\lambda}\right)^k\cdot\left(1-\frac{\lambda}{n}\right)^n$$

$$= \frac{n!}{(n-k)!\,(n-\lambda)^k} \cdot \frac{\lambda^k}{k!} \cdot \left(1 - \frac{\lambda}{n}\right)^n .$$

Equations (2) and (3) now yield

$$\lim_{n\to\infty} \left[\binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \left(\frac{\lambda}{n}\right)\right)^{n-k} \right] = 1 \cdot \frac{\lambda^k}{k!} \cdot e^{-\lambda} \; = \; e^{-\lambda} \cdot \frac{\lambda^k}{k!},$$

which proves equation (1). ■