CSE 107
Fall 2022
Midterm Exam 2 **Solutions**

1. (20 Points) Bob throws a dart at a circular target of radius r. He hits the target with certainty, but is equally likely to hit any point within the target. Let Z be the distance from Bob's dart to the center of the target.

   a. (10 Points) Find the CDF $F_Z(z)$ and the PDF $f_Z(z)$.
   Solution:
   Since Bob is equally likely to hit any point, the probability of the dart landing within any region of the target is proportional to the area of that region. Since the area of the target is πr^2 we have

   $$P(\text{dart lands in region } A) = \frac{\text{area}(A)}{\pi r^2}$$

   Therefore, for any z in $0 \le z \le r$, we have

   $$P(Z \le z) = \frac{\pi z^2}{\pi r^2} = \frac{z^2}{r^2}$$

   and hence

   $$F_Z(z) = \begin{cases} \mathbf{0} & \textbf{if } \boldsymbol{z < 0} \\ \boldsymbol{\frac{z^2}{r^2}} & \textbf{if } \boldsymbol{0 \le z \le r} \\ \mathbf{1} & \textbf{if } \boldsymbol{z > r} \end{cases}$$

   Differentiating with respect to z gives us

   $$f_Z(z) = \begin{cases} \boldsymbol{\frac{2z}{r^2}} & \textbf{if } \boldsymbol{0 \le z \le r} \\ \mathbf{0} & \textbf{otherwise} \end{cases}$$

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   b. (5 Points) Find the mean $E[Z]$.
   Solution:

   $$E[Z] = \int_{-\infty}^{\infty} z f_Z(z)\, dz = \int_0^r \frac{2z^2}{r^2}\, dz = \frac{2}{r^2} \cdot \frac{z^3}{3}\bigg|_0^r = \frac{2r^3}{3r^2} = \boldsymbol{\frac{2r}{3}}$$

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   c. (5 Points) Find the variance $Var(Z)$.
   Solution:

   $$E[Z^2] = \int_0^r \frac{2z^3}{r^2}\, dz = \frac{2}{r^2} \cdot \frac{z^4}{4}\bigg|_0^r = \frac{r^2}{2}$$

   and hence

   $$Var(Z) = E[Z^2] - E[Z]^2 = \frac{r^2}{2} - \left(\frac{2r}{3}\right)^2 = \left(\frac{1}{2} - \frac{4}{9}\right) r^2 = \boldsymbol{\frac{r^2}{18}}$$

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2. (20 Points) A city's temperature in degrees Celsius is modeled as a normal random variable X with mean 10 and standard deviation 10. Let Y be its temperature in Fahrenheit, where X and Y are related by

$$X = \frac{5(Y-32)}{9}.$$

What is the probability that the temperature is above 77 degrees Fahrenheit?

Solution:

$$P(Y > 77) = P\left(X > \frac{5(77-32)}{9}\right)$$

$$= P(X > 25)$$

$$= P\left(\frac{X-10}{10} > \frac{25-10}{10}\right)$$

$$= P\left(\frac{X-10}{10} > 1.5\right)$$

$$= 1 - P\left(\frac{X-10}{10} \leq 1.5\right)$$

$$= 1 - \Phi(1.5)$$

$$= 1 - 0.9332$$

$$= \mathbf{0.0668}$$

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Alternate Solution:

We have $Y = (9X + 160)/5$, and therefore Y is normal with mean $(9 \cdot 10 + 160)/5 = 50$, and standard deviation $(9/5) \cdot 10 = 18$. Thus

$$P(Y > 77) = P\left(\frac{Y-50}{18} > \frac{77-50}{18}\right)$$

$$= P\left(\frac{Y-50}{18} > 1.5\right)$$

$$= 1 - P\left(\frac{Y-50}{18} \leq 1.5\right)$$

$$= 1 - \Phi(1.5)$$

$$= \mathbf{0.0668}$$

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3. (20 Points) Let X and Y be jointly continuous random variables, and suppose

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{1}{y} & \text{if } 0 < y \leq 1 \text{ and } 1 - y \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_Y(y) = \begin{cases} 2y & \text{if } 0 < y \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

Hint: Before you do the following problems, draw a picture of the region defined by the inequalities $0 < y \leq 1$ and $1 - y \leq x \leq 1$.

a. (5 Points) Determine the joint PDF $f_{X,Y}(x,y)$.
Solution:
The region defined by the inequalities $0 < y \leq 1$ and $1 - y \leq x \leq 1$ is the triangle with vertices $(0,1)$, $(1,0)$ and $(1,1)$, which has area $1/2$. Therefore

$$f_{X,Y}(x,y) = \begin{cases} \mathbf{2} & \textbf{if } \mathbf{0 < y \leq 1} \textbf{ and } \mathbf{1 - y \leq x \leq 1} \\ \mathbf{0} & \textbf{otherwise} \end{cases}$$

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b. (5 Points) Determine the marginal PDF $f_X(x)$.
Solution:
For any x in the range $0 \leq x \leq 1$ we have

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \int_{1-x}^{1} 2\, dy = 2y|_{1-x}^{1} = 2\big(1 - (1-x)\big) = 2x$$

so

$$f_X(x) = \begin{cases} \mathbf{2x} & \textbf{if } \mathbf{0 \leq x \leq 1} \\ \mathbf{0} & \textbf{otherwise} \end{cases}$$

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c. (5 Points) Determine the expected value $E[X]$.
Solution:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^1 2x^2\, dx = \frac{2}{3}x^3\Big|_0^1 = \mathbf{\frac{2}{3}}$$

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d. (5 Points) Determine the conditional expectation $E[X|Y=y]$.
Solution:

$$E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx = \int_{1-y}^{1} \frac{x}{y}\, dx = \frac{1}{y} \cdot \frac{x^2}{2}\Big|_{1-y}^{1}$$

$$= \mathbf{\frac{1-(1-y)^2}{2y}} = \mathbf{\frac{2y - y^2}{2y}} = \mathbf{\frac{2-y}{2}} = \mathbf{1 - \frac{y}{2}}$$

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4. (20 Points) Let X be an exponential random variable with parameter λ, and let $Y = X + 1$. Determine the PDF $f_Y(y)$.

Solution:
Using the formula for the PDF of $Y = aX + b$ derived in class

$$f_{Y(y)} = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

with $a = b = 1$, we have

$$f_{Y(y)} = f_X(y-1)$$

$$= \begin{cases} \boldsymbol{\lambda e^{-\lambda(y-1)}} & \textbf{if } \boldsymbol{y-1 \geq 0} \\ \boldsymbol{0} & \boldsymbol{y-1 < 0} \end{cases}$$

$$= \begin{cases} \boldsymbol{\lambda e^{\lambda} \cdot e^{-\lambda y}} & \textbf{if } \boldsymbol{y \geq 1} \\ \boldsymbol{0} & \boldsymbol{y < 1} \end{cases}$$

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Alternate Solution:
Using the two-step process, we have

$$F_Y(y) = P(Y \leq y) = P(X + 1 \leq y) = P(X \leq y - 1) = F_X(y-1)$$

Differentiating with respect to y gives

$$f_Y(y) = f_X(y-1)$$

$$= \begin{cases} \boldsymbol{\lambda e^{-\lambda(y-1)}} & \textbf{if } \boldsymbol{y-1 \geq 0} \\ \boldsymbol{0} & \boldsymbol{y-1 < 0} \end{cases}$$

$$= \begin{cases} \boldsymbol{\lambda e^{\lambda} \cdot e^{-\lambda y}} & \textbf{if } \boldsymbol{y \geq 1} \\ \boldsymbol{0} & \boldsymbol{y < 1} \end{cases}$$

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5. (20 Points) Alice is at the casino again, with a choice of two games. The first returns winnings (positive or negative) that are normally distributed with parameters $\mu = 1$ and $\sigma = 2$. The second is uniformly distributed with winnings in the range -1 to 2. (All amounts are in dollars.) She flips a coin with $P(\text{head}) = p$ to decide which game to play. If heads, she plays the first game, and if tails, she plays the second. Determine her expected winnings, in terms of p.

Solution:

Let X be the amount of money Alice wins, and let A be the event that the coin flip is heads. We are given that $P(A) = p$ and $P(A^c) = 1 - p$. We also know that $E[X|A] = 1$ and $E[X|A^c] = 1/2$. The total expectation theorem now gives

$$E[X] = pE[X|A] + (1-p)E[X|A^c]$$

$$= p \cdot 1 + (1-p) \cdot \frac{1}{2}$$

$$= \boldsymbol{\frac{p+1}{2}}$$

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