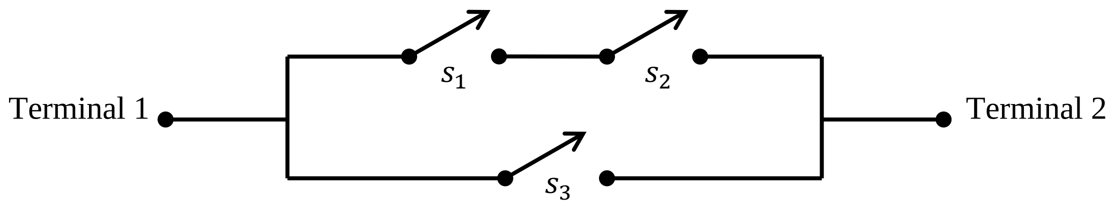
CSE 107
Fall 2022
Midterm Exam 1 Solutions

1. (20 Points) In the circuit diagram below, switches s_1, s_2 and s_3 are randomly and independently set in the open or closed state. Let A_i be the event that s_i is **open** (for $i = 1, 2, 3$), and let A be the event that there is a **closed path** from terminal 1 to terminal 2.



Suppose that $P(A_i) = p_i$, for $i = 1, 2, 3$. Determine $P(A)$ in terms of p_1, p_2 and p_3.

Solution:
There is a **closed path** from terminal 1 to terminal 2 if and only if either s_1 and s_2 are closed, **or** s_3 is closed. Therefore by independence

$$P(A) = P\big((A_1^c \cap A_2^c) \cup A_3^c\big)$$

$$= P(A_1^c \cap A_2^c) + P(A_3^c) - P(A_1^c \cap A_2^c \cap A_3^c)$$

$$= (\mathbf{1} - \boldsymbol{p_1})(\mathbf{1} - \boldsymbol{p_2}) + (\mathbf{1} - \boldsymbol{p_3}) - (\mathbf{1} - \boldsymbol{p_1})(\mathbf{1} - \boldsymbol{p_2})(\mathbf{1} - \boldsymbol{p_3}).$$

Alternate Solution:
There is **no closed path** from terminal 1 to terminal 2 if and only if both s_3 is open, **and** either s_1 or s_2 are open. Thus, again by independence,

$$P(A) = 1 - P(A^c) = 1 - P\big((A_1 \cup A_2) \cap A_3\big)$$

$$= 1 - P(A_1 \cup A_2)P(A_3)$$

$$= 1 - \big(P(A_1) + P(A_2) - P(A_1 \cap A_2)\big)P(A_3)$$

$$= \mathbf{1} - (\boldsymbol{p_1} + \boldsymbol{p_2} - \boldsymbol{p_1}\boldsymbol{p_2})\boldsymbol{p_3}$$

$$= \mathbf{1} - \boldsymbol{p_1}\boldsymbol{p_3} - \boldsymbol{p_2}\boldsymbol{p_3} + \boldsymbol{p_1}\boldsymbol{p_2}\boldsymbol{p_3}.$$

2. (20 Points) A system consists of n identical components, each of which is operational with probability p, independent of other components. The system is operational if at least m out of the n components are operational. What is the probability that the system is operational?

Solution:
Let X be the number of components that are operational. Then X is a Binomial random variable with parameters n and p, and its PMF is

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad \text{for } k = 0, 1, 2, \dots, n.$$

The probability that m or more of the n components are operational is therefore

$$P(X \geq m) = \sum_{k=m}^{n} p_X(k) = \boldsymbol{\sum_{k=m}^{n} \binom{n}{k} p^k (1-p)^{n-k}}.$$

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3. (20 Points) Alice and Bob have a chess match in which the first player to win a game wins the match. Each game has one of 3 possible outcomes: Bob wins, Alice wins, or the game is a draw. One game is played each day until someone wins, so the match is of potentially unlimited duration. The prize money starts at \$100 and goes up by \$100 each day a match is played. Alice wins with probability 0.4, Bob wins with probability 0.3, and a draw occurs with probability 0.3.

a. (10 Points) What is the probability that Alice wins the match?

Solution:
Let A_k be the event that Alice wins the match on the k^{th} game, and let A be the event that Alice wins the match. Then $P(A_k) = (.3)^{k-1}(.4)$, and hence

$$P(A) = \sum_{k=1}^{\infty} (.3)^{k-1}(.4) = (.4) \cdot \sum_{k=1}^{\infty} (.3)^{k-1} = (.4) \cdot \sum_{k=0}^{\infty} (.3)^k = \frac{.4}{1-(.3)} = \boldsymbol{\frac{4}{7}}$$

b. (10 Points) Determine the mean and standard deviation of the total prize money.

Solution:
Let X be the duration of the match, in days. The probability that a particular game is won by somebody is $0.3 + 0.4 = 0.7$, so that X is a Geometric random variable with parameter 0.7. The total prize money is $100X$, which has mean

$$E[100X] = 100 \cdot E[X] = \boldsymbol{\frac{100}{0.7}} = \$\boldsymbol{142.86},$$

variance $Var(X) = 100^2 \cdot Var(X) = 100^2 \cdot \frac{1-(0.7)}{(0.7)^2} = 100^2 \cdot (.612244)$, and standard deviation

$$\sigma_X = \sqrt{Var(X)} = 100 \cdot (.78246) = \$\boldsymbol{78.25}$$

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4. (20 Points) A 3-sided die and a coin, which are neither fair nor independent, are rolled and tossed, respectively. The die has faces $\{1, 2, 3\}$ and the coin has sides labeled $\{1, 2\}$. Let X be the outcome of the die, and Y the outcome of the coin. The *conditional* PMF $p_{X|Y}(x|y)$ is given by the following table.

y \ x	1	2	3
1	$2/8$	$5/8$	$1/8$
2	$1/8$	$3/8$	$4/8$

Also, the *marginal* PMF $p_Y(y)$ is given by the following table.

y	
1	$1/3$
2	$2/3$

a. (5 Points) Fill in the following table giving the *joint* PMF $p_{X,Y}(x, y) = \boldsymbol{p_Y(y) \cdot p_{X|Y}(x|y)}$

Solution:

y \ x	1	2	3
1	$\mathbf{2/24}$	$\mathbf{5/24}$	$\mathbf{1/24}$
2	$\mathbf{2/24}$	$\mathbf{6/24}$	$\mathbf{8/24}$

b. (5 Points) Fill in the following table giving the *marginal* PMF $p_X(x) = \sum_y \boldsymbol{p_{X,Y}(x, y)}$

Solution:

x	1	2	3
$p_X(x)$	$\mathbf{4/24}$	$\mathbf{11/24}$	$\mathbf{9/24}$

c. (5 Points) Fill in the following table giving the conditional PMF $p_{Y|X}(y|x) = \frac{\boldsymbol{p_{X,Y}(x,y)}}{\boldsymbol{p_X(x)}}$

Solution:

y \ x	1	2	3
1	$\mathbf{1/2}$	$\mathbf{5/11}$	$\mathbf{1/9}$
2	$\mathbf{1/2}$	$\mathbf{6/11}$	$\mathbf{8/9}$

d. (5 Points) Given that the coin flip is 2, what is the probability that the die roll is 3?

Solution: $p_{X|Y}(3|2) = \mathbf{4/8 = 1/2}$

5. (20 Points) The number X of phone calls received by a call center within a certain time period is a Poisson random variable with parameter λ. Determine the smallest positive number λ such that the probability of receiving at least one call is at least $1/2$.

Solution:
We require that $P(X \geq 1) \geq 1/2$. Hence

$$1 - P(X = 0) \geq \frac{1}{2} \quad \Longrightarrow \quad 1 - \frac{1}{2} \geq P(X = 0) \quad \Longrightarrow \quad p_X(0) \leq \frac{1}{2},$$

and therefore

$$e^{-\lambda} \cdot \frac{\lambda^0}{0!} = e^{-\lambda} \leq \frac{1}{2}$$

$$\therefore \qquad -\lambda \leq \ln\left(\frac{1}{2}\right) = -\ln(2)$$

$$\therefore \qquad \lambda \geq \ln(2)$$

The smallest such λ is $\lambda = \mathbf{ln(2) = 0.6931}$. ∎