CSE 107
Homework Assignment 7
Solutions

Section 3.5: Conditioning

1. Let X have a uniform distribution in the unit interval $[0, 1]$, and let Y have an exponential distribution with parameter $\nu = 2$. Assume that X and Y are independent, and let $Z = X + Y$.

   a. Find $P(Y \geq X)$.
   Solution:
   We are given
   $$f_X(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$
   and
   $$f_Y(y) = \begin{cases} 2e^{-2y} & \text{if } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

   Since X and Y are independent
   $$f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y) = \begin{cases} 2e^{-2y} & \text{if } 0 \leq x \leq 1 \text{ and } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

   Therefore
   $$P(Y \geq X) = \int_0^1 \int_x^\infty 2e^{-2y}\, dy\, dx$$
   $$= \int_0^1 -e^{-2y}|_x^\infty\, dx$$
   $$= \int_0^1 e^{-2x}\, dx$$
   $$= -\frac{1}{2} e^{-2x}\Big|_0^1 = -\frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{e^{-2}} - \mathbf{1}) = \frac{\mathbf{1} - \boldsymbol{e^{-2}}}{\mathbf{2}}$$

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   b. Find the conditional PDF of Z given that $Y = y$.
   Solution:
   $$F_{Z|Y}(z|y) = P(Z \leq z | Y = y)$$
   $$= P(X + Y \leq z \mid Y = y)$$
   $$= P(X + y \leq z \mid Y = y)$$
   $$= P(X \leq z - y \mid Y = y)$$

$$= P(X \le z - y)$$

$$= F_X(z - y)$$

Where the second to last step follows from the independence of X and Y. Upon differentiating both sizes with respect to z, we obtain

$$f_{Z|Y}(z|y) = f_X(z - y)$$

$$= \begin{cases} \mathbf{1} & \textbf{if } \mathbf{0 \le z - y \le 1} \\ \mathbf{0} & \textbf{otherwise} \end{cases}$$

$$= \begin{cases} \mathbf{1} & \textbf{if } \boldsymbol{y \le z \le y + 1} \\ \mathbf{0} & \textbf{otherwise} \end{cases}$$

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c. Find the conditional PDF of Y given that $Z = 3$.
Solution:
By the continuous Bayes' rule

$$f_{Y|Z}(y|z) = \frac{f_{Z|Y}(z|y) \cdot f_Y(y)}{f_Z(z)}$$

so that

$$f_{Y|Z}(y|3) = \frac{f_{Z|Y}(3|y) \cdot f_Y(y)}{f_Z(3)}$$

We have the numerator by part (b). To compute the denominator, note that for $z = 3$, the function $f_{Z|Y}(z|y)$ is non-zero only if $0 \le 3 - y \le 1$, which implies $2 \le y \le 3$. Therefore

$$f_Z(3) = \int_2^3 f_{Z|Y}(3|y) \cdot f_Y(y)\, dy$$

$$= \int_2^3 1 \cdot 2e^{-2y}\, dy$$

$$= -e^{-2y}|_2^3$$

$$= -(e^{-6} - e^{-4}) = e^{-4} - e^{-6}$$

and therefore

$$f_{Y|Z}(y|3) = \begin{cases} \dfrac{\mathbf{2}\boldsymbol{e^{-2y}}}{\boldsymbol{e^{-4} - e^{-6}}} & \textbf{if } \mathbf{2 \le \boldsymbol{y} \le 3} \\ \mathbf{0} & \textbf{otherwise} \end{cases}$$

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Section 4.1: Derived Distributions

2. Let X be a random variable with PDF $f_X(x)$. Find the PDF of the random variable $Y = |X|$ in the following three cases.

   a. X is exponentially distributed with parameter λ.

   Solution:

   We are given

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

   Since $P(X < 0) = 0$, the random variable X assumes only non-negative values, and hence the distribution of $Y = |X|$ is identical to that of X.

$$f_Y(y) = \begin{cases} \boldsymbol{\lambda e^{-\lambda y}} & \textbf{if } \boldsymbol{y \geq 0} \\ \mathbf{0} & \textbf{if } \boldsymbol{y < 0} \end{cases}$$

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   b. X is uniformly distributed in the interval $[-1, 2]$.

   Solution:

   We have $P(-1 \leq X \leq 1) = 2P(1 \leq X \leq 2)$, hence $P(0 \leq Y \leq 1) = 2P(1 \leq Y \leq 2)$. Also $P(0 \leq Y \leq 1) + P(1 \leq Y \leq 2) = 1$, so that $P(0 \leq Y \leq 1) = 2/3$ and $P(1 \leq Y \leq 2) = 1/3$. Thus

$$f_Y(y) = \begin{cases} \mathbf{\frac{2}{3}} & \textbf{if } \boldsymbol{0 \leq y \leq 1} \\ \mathbf{\frac{1}{3}} & \textbf{if } \boldsymbol{1 \leq y \leq 2} \\ \mathbf{0} & \textbf{otherwise} \end{cases}$$

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   c. $f_X(x)$ is a general PDF.

   Solution:

   Using the 2-step process, we have

$$F_Y(y) = P(Y \leq y) = P(|X| \leq y) = P(-y \leq X < y) = F_X(y) - F_X(-y)$$

   Differentiating both sides with respect to y gives $f_Y(y) = f_X(y) + f_X(-y)$, and hence

$$f_Y(y) = \begin{cases} \boldsymbol{f_X(y) + f_X(-y)} & \textbf{if } \boldsymbol{y \geq 0} \\ \mathbf{0} & \textbf{otherwise} \end{cases}$$

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3. Let X and Y be independent discrete random variables with the same PMF:

$$p_X(x) = \begin{cases} 1/4 & \text{if } x = 1 \\ 1/4 & \text{if } x = 2 \\ 1/2 & \text{if } x = 3 \\ 0 & \text{otherwise} \end{cases}$$

and similarly for Y. Use the convolution product to find the PMF of $Z = X + Y$.

Solution:

The convolution product gives

$$p_Z(z) = \sum_x p_X(x) p_Y(z - x)$$

$$= p_X(1)p_Y(z-1) + p_X(2)p_Y(z-2) + p_X(3)p_Y(z-3)$$

$$= \frac{1}{4} \cdot p_Y(z-1) + \frac{1}{4} \cdot p_Y(z-2) + \frac{1}{2} \cdot p_Y(z-3)$$

The function $p_Z(z)$ will be non-zero only for z in $\{1,2,3\} + \{1,2,3\} = \{2,3,4,5,6\}$. We compute

$$p_Z(2) = \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot 0 = \frac{1}{16}$$

$$p_Z(3) = \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{2} \cdot 0 = \frac{1}{8}$$

$$p_Z(4) = \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} = \frac{5}{16}$$

$$p_Z(5) = \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{4}$$

$$p_Z(6) = \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

Therefore

$$p_Z(z) = \begin{cases} \mathbf{\frac{1}{16}} & \textbf{if } \boldsymbol{z = 2} \\ \mathbf{\frac{1}{8}} & \textbf{if } \boldsymbol{z = 3} \\ \mathbf{\frac{5}{16}} & \textbf{if } \boldsymbol{z = 4} \\ \mathbf{\frac{1}{4}} & \textbf{if } \boldsymbol{z = 5, 6} \\ \mathbf{0} & \textbf{otherwise} \end{cases}$$

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4. Let X be continuous uniform on the interval $[0, 2]$, and Y continuous uniform on $[3, 4]$. Assume that X and Y are independent. Use the convolution product to find the PDF of $Z = X + Y$.
Solution:
We have

$$f_X(x) = \begin{cases} \frac{1}{2} & \text{if } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_Y(y) = \begin{cases} 1 & \text{if } 3 \le y \le 4 \\ 0 & \text{otherwise} \end{cases}$$

The convolution product gives

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z - x)\, dx$$

which has a non-zero integrand if and only if both $0 \le x \le 2$ and $3 \le z - x \le 4$. This is equivalent to both $0 \le x \le 2$ and $z - 4 \le x \le z - 3$, which implies

$$\max(0, z - 4) \le x \le \min(2, z - 3).$$

Therefore

$$f_Z(z) = \int_{\max(0,z-4)}^{\min(2,z-3)} \frac{1}{2} \cdot 1\, dx$$

$$= \begin{cases} \boldsymbol{\dfrac{\min(2, z-3) - \max(0, z-4)}{2}} & \textbf{if } \boldsymbol{3 \le z \le 6} \\ \mathbf{0} & \textbf{otherwise} \end{cases}$$

$$= \begin{cases} \boldsymbol{\dfrac{z-3}{2}} & \textbf{if } \boldsymbol{3 \le z \le 4} \\ \boldsymbol{\dfrac{1}{2}} & \textbf{if } \boldsymbol{4 \le z \le 5} \\ \boldsymbol{\dfrac{6-z}{2}} & \textbf{if } \boldsymbol{5 \le z \le 6} \\ \mathbf{0} & \textbf{otherwise} \end{cases}$$

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