CSE 107
Homework Assignment 6
Solutions to Selected Problems

Section 3.3: Normal Random Variables

1. Let X be normal with mean 1 and variance 4, and let $Y = 2X + 3$.

   a. Calculate the PDF of Y.
   Solution:
   Y is normal with mean $\mu = 2 \cdot 1 + 3 = 5$ and variance $\sigma^2 = 2^2 \cdot 4 = 16$, and therefore

$$f_Y(y) = \frac{1}{4\sqrt{2\pi}} \cdot e^{-(y-5)^2/32}$$

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   b. Find $P(Y \geq 0)$.
   Solution:

$$P(Y \geq 0) = P\left(\frac{Y-5}{4} \geq \frac{0-5}{4}\right)$$

$$= P\left(\frac{Y-5}{4} \geq -\frac{5}{4}\right)$$

$$= P\left(\frac{Y-5}{4} \leq \frac{5}{4}\right) \qquad \text{(by symmetry)}$$

$$= \Phi(1.25) = \mathbf{0.8944}$$

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2. A signal of amplitude $s = 2$ is transmitted from a satellite and is corrupted by noise. The received signal is $X = s + W$, where the noise W is normal. When the weather is good, W has mean 0 and variance 1. When the weather is bad, W has mean 0 and variance 4. Good and bad weather are equally likely. In the absence of any weather information, determine the following.

   a. The PDF of X.
   Solution:
   Let G and B be the events good weather and bad weather, respectively. We are given that $P(G) = P(B) = 1/2$. We first calculate the PDF of W using the total probability theorem for densities.

$$f_W(w) = P(G) \cdot f_{W|G}(w) + P(B) \cdot f_{W|B}(w)$$

$$= \frac{1}{2} \cdot \frac{e^{-w^2/2}}{\sqrt{2\pi}} + \frac{1}{2} \cdot \frac{e^{-w^2/8}}{\sqrt{2\pi} \cdot 2}$$

$$= \frac{1}{2}\left(\frac{e^{-w^2/2}}{\sqrt{2\pi}} + \frac{e^{-w^2/8}}{\sqrt{2\pi} \cdot 2}\right)$$

   To find the PDF of $X = 2 + W$, we do the change of variable $w = x - 2$.

$$f_X(x) = f_W(x-2) = \frac{\mathbf{1}}{\mathbf{2}}\left(\frac{\boldsymbol{e^{-(x-2)^2/2}}}{\boldsymbol{\sqrt{2\pi}}} + \frac{\boldsymbol{e^{-(x-2)^2/8}}}{\boldsymbol{\sqrt{2\pi}\cdot 2}}\right)$$

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b. The probability that X is between 1 and 3.
Solution:

$$P(1 \le X \le 3) = \int_1^3 \frac{1}{2}\left(\frac{e^{-(x-2)^2/2}}{\sqrt{2\pi}} + \frac{e^{-(x-2)^2/8}}{\sqrt{2\pi}\cdot 2}\right) dx$$

$$= \frac{1}{2}\left(\int_1^3 \frac{e^{-\left(\frac{x-2}{1}\right)^2/2}}{\sqrt{2\pi}} dx + \int_1^3 \frac{e^{-\left(\frac{x-2}{2}\right)^2/2}}{\sqrt{2\pi}\cdot 2} dx\right)$$

$$= \frac{1}{2}\left(\int_{-1}^1 \frac{e^{-u^2/2}}{\sqrt{2\pi}} du + \int_{-1/2}^{1/2} \frac{e^{-v^2/2}}{\sqrt{2\pi}}\, dv\right) \quad \text{using } u = x-2, \text{and } v = \frac{x-2}{2}$$

$$= \frac{1}{2}\left(\Phi(1) - \Phi(-1) + \Phi\left(\frac{1}{2}\right) - \Phi\left(-\frac{1}{2}\right)\right)$$

$$= \frac{1}{2}\left(\Phi(1) - \big(1-\Phi(1)\big) + \Phi\left(\frac{1}{2}\right) - \left(1 - \Phi\left(\frac{1}{2}\right)\right)\right)$$

$$= \frac{1}{2}\left(2\Phi(1) + 2\Phi\left(\frac{1}{2}\right) - 2\right)$$

$$= \Phi(1) + \Phi\left(\frac{1}{2}\right) - 1 = \mathbf{0.5328}$$

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3. A transmitter sends the signals -1 and $+1$, with probabilities p and $1-p$, respectively. The communication channel introduces additive noise that is normal with mean 0 and variance σ^2. This noise is independent of the particular signal sent. The receiver at the other end of the channel receives a signal which is the sum of the transmitted signal and the channel noise.

   a. Let a be a constant between -1 and $+1$. The receiver interpretes its signal as -1 if it is less than a, and as $+1$ if it is greater than a. Find a formula (in terms of a, p and σ) for the probability that the receiver makes an error.

   Solution:
   Let $s \in \{-1, +1\}$ be the signal and X be a normal random variable with mean 0 and variance σ^2 representing the noise. Then the received signal is $s + X$. There are two ways an error can occur.

   (1) $s = -1$ but $s + X > a$, so it is interpreted as $+1$. In this case $P(-1 + X > a) = P(X > a+1) = P\left(\frac{X-0}{\sigma} > \frac{a+1}{\sigma}\right) = 1 - P\left(\frac{X-0}{\sigma} \le \frac{a+1}{\sigma}\right) = 1 - \Phi\left(\frac{a+1}{\sigma}\right)$.

(2) $s = +1$ but $s + X < a$, so it is interpreted as -1. Then we have $P(1 + X < a) = P(X < a - 1) = P\left(\frac{X-0}{\sigma} < \frac{a-1}{\sigma}\right) = \Phi\left(\frac{a-1}{\sigma}\right)$.

Therefore, by the total probability theorem

$$P(\text{error}) = \boldsymbol{p}\left(\mathbf{1} - \boldsymbol{\Phi}\left(\frac{\boldsymbol{a}+\mathbf{1}}{\boldsymbol{\sigma}}\right)\right) + (\mathbf{1} - \boldsymbol{p})\boldsymbol{\Phi}\left(\frac{\boldsymbol{a}-\mathbf{1}}{\boldsymbol{\sigma}}\right).$$

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b. Determine the probability that the receiver makes an error if $p = 2/5$, $a = 1/2$ and $\sigma^2 = 1/4$.

Solution:
We have

$$P(\text{error}) = \frac{2}{5}\cdot\left(1 - \Phi\left(\frac{3/2}{1/2}\right)\right) + \frac{3}{5}\cdot\Phi\left(\frac{-1/2}{1/2}\right)$$

$$= \frac{2}{5}\cdot\big(1 - \Phi(3)\big) + \frac{3}{5}\cdot\big(1 - \Phi(1)\big)$$

$$= (.4)\cdot(1 - .9987) + (.6)\cdot(1 - .8413)$$

$$= \mathbf{0.09574}$$

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Section 3.4: Multiple Continuous Random Variables

4. An old modem can take anywhere from 0 to 30 seconds to establish a connection, with all times between 0 and 30 being equally likely.

a. What is the probability that if you use this modem you will have to wait more than 15 seconds to connect?
Solution:
We are given that the time T to make a connection is uniform, so

$$f_T(t) = \begin{cases} \dfrac{1}{30} & \text{if } 0 \le t \le 30 \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$P(T \ge 15) = \int_{15}^{\infty} f_T(t)\,dt = \int_{15}^{30} \frac{1}{30}\,dt = \mathbf{\frac{1}{2}}$$

which could have been seen from the symmetry of the uniform distribution. ■

b. Given that you have already waited 10 seconds, what is the probability of having to wait at least 10 more seconds?
Solution:
We seek $P(T > 20 | T > 10)$. First, we determine $f_{T|T>10}(t)$. By the formula at the bottom of p.11 in lecture notes for 11-3-22, we have

$$f_{T|T>10}(t) = \begin{cases} \dfrac{f_T(t)}{P(T > 10)} & \text{if } 10 \leq t \leq 30 \\ 0 & \text{otherwise} \end{cases}$$

Observe that

$$P(T > 10) = \int_{10}^{30} \frac{1}{30}\, dt = \frac{30 - 10}{30} = 2/3$$

and therefore

$$f_{T|T>10}(t) = \begin{cases} \dfrac{1/30}{2/3} & \text{if } 10 \leq t \leq 30 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \dfrac{1}{20} & \text{if } 10 \leq t \leq 30 \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$P(T > 20|T > 10) = \int_{20}^{30} \frac{1}{20}\, dt = \frac{30 - 20}{20} = \mathbf{\frac{1}{2}}$$

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5. Consider a random variable X with PDF

$$f_X(x) = \begin{cases} \dfrac{2x}{3} & \text{if } 1 < x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

and let A be the event $A = \{X \geq 1.5\}$. Determine $E[X]$, $P(A)$ and $E[X|A]$.
Solution:
We have

$$E[X] = \int_{-\infty}^{\infty} f_X(x)\, dx = \int_{1}^{2} \frac{2x^2}{3}\, dx = \frac{2}{3} \cdot \frac{x^3}{3}\bigg|_1^2 = \frac{2(8-1)}{9} = \mathbf{\frac{14}{9}}$$

and

$$P(A) = \int_{3/2}^{2} \frac{2x}{3}\, dx = \frac{2}{3} \cdot \frac{x^2}{2}\bigg|_{3/2}^{2} = \frac{4 - \frac{9}{4}}{3} = \mathbf{\frac{7}{12}}$$

The conditional PDF $f_{X|A}(x)$ is given by

$$f_{X|A}(x) = \begin{cases} \dfrac{f_X(x)}{P(A)} & \text{if } 1.5 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{2x/3}{7/12} & \text{if } 1.5 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{8x}{7} & \text{if } 1.5 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

and therefore

$$
\begin{aligned}
E[X|A] &= \int_{-\infty}^{\infty} x f_{X|A}(x)\,dx \\
&= \int_{3/2}^{2} \frac{8}{7} \cdot x^2\,dx \\
&= \frac{8}{7} \cdot \frac{x^3}{3}\bigg|_{3/2}^{2} \\
&= \frac{8}{21}\left(8 - \frac{27}{8}\right) \\
&= \frac{64-27}{21} = \mathbf{\frac{37}{21}}
\end{aligned}
$$

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Section 3.5: Conditioning

6. Alice is vacationing in Monte Carlo. The amount X of money she takes to the casino each evening is a random variable with PDF of the form

$$
f_X(x) = \begin{cases} ax & \text{if } 0 \le x \le 40 \\ 0 & \text{otherwise} \end{cases}
$$

At the end of each night of gambling, the amount Y that she takes back to her hotel room is uniformily distributed between 0 and twice the amount she came with.

a. Find the value of a in the above formula for $f_X(x)$.

Solution:
We have

$$
1 = \int_0^{40} ax\,dx = \frac{1}{2}ax^2\bigg|_0^{40} = a \cdot 800
$$

so that $a = \mathbf{1/800}$.

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b. Write down the conditional PDF $f_{Y|X}(y|x)$.

Solution:
We are given that $f_{Y|X}(y|x)$ is uniform on the interval $0 \le y \le 2x$, so that

$$
f_{Y|X}(y|x) = \begin{cases} \mathbf{\dfrac{1}{2x}} & \mathbf{\text{if } 0 \le y \le 2x} \\ \mathbf{0} & \mathbf{\text{otherwise}} \end{cases}
$$

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c. Determine the joint PDF $f_{X,Y}(x,y)$.

Solution:

$$f_{X,Y}(x,y) = f_{Y|X}(y|x) \cdot f_X(x)$$

$$= \frac{1}{2x} \cdot \frac{x}{800}$$

$$= \begin{cases} \boldsymbol{\frac{1}{1600}} & \textbf{if } \boldsymbol{0 \leq x \leq 40} \textbf{ and } \boldsymbol{0 \leq y \leq 2x} \\ \boldsymbol{0} & \textbf{otherwise} \end{cases}$$

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d. Determine the marginal PDF $f_Y(y)$.

Solution:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

$$= \int_{\frac{y}{2}}^{40} \frac{1}{1600} dx$$

$$= \begin{cases} \boldsymbol{\frac{80-y}{3200}} & \textbf{if } \boldsymbol{0 \leq y \leq 80} \\ \boldsymbol{0} & \textbf{otherwise} \end{cases}$$

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e. Find the expected value of her profit $Z = Y - X$.

Solution:
Symmetry alone suggests that $E[Z]$ should be zero, since on any given night Alice is equally likely to be a winner as loser. We compute

$$E[X] = \int_0^{40} x \cdot \frac{x}{800} dx = \frac{x^3}{2400}\bigg|_0^{40} = \frac{80}{3}$$

and

$$E[Y] = \int_0^{80} y \cdot \frac{80-y}{3200} dy = \frac{40y^2 - \frac{1}{3}y^3}{3200}\Bigg|_0^{80} = \frac{80}{3}$$

so that

$$E[Z] = E[Y-X] = E[X] - E[Y] = \mathbf{0}$$

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f. What is the probability that on a given night, Alice makes a positive profit at the casino?

Solution:
Again symmetry suggests that $P(Z > 0) = P(Z < 0) = \mathbf{1/2}$. We can use the joint PDF $f_{X,Y}(x, y)$ to compute

$$P(Z > 0) = P(Y > X)$$

$$= \iint_{\{(x,y)\,|\,y>x\}} \frac{1}{1600}\, dy\, dx$$

$$= \int_0^{40} \int_x^{2x} \frac{1}{1600}\, dy\, dx$$

$$= \int_0^{40} \frac{2x - x}{1600}\, dx$$

$$= \int_0^{40} \frac{x}{1600}\, dx$$

$$= \left.\frac{x^2}{3200}\right|_0^{40} = \frac{1600}{3200} = \mathbf{\frac{1}{2}}$$

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