CSE 107
Homework Assignment 5
Solutions

Section 2.7: Independence

1. This problem is a continuation of problem 2 from hw4, and utilizes the answers to parts (a), (b), and (c) of that problem.

   Joe Lucky plays the lottery on any given week with probability p, independently of whether he played on any other week. Each time he plays, he has a probability q of winning, again independently of everything else. During a fixed time period of n weeks, let X be the number of weeks that he played the lottery and Y be the number of weeks that he won.

   d. Find the marginal PMF $p_Y(y)$. Hint: The standard approach would be to take the answer to (c) and sum over x. This will give the correct answer, but with much algebra. Instead, try to see why Y is a binomial random variable. What are its parameters?

   Solution:
   In any given week, Joe wins with probability pq, as seen from the tree diagram in the solution to part (a). Therefore Y is binomial with parameters n and pq, hence

$$p_Y(y) = \binom{n}{y}(pq)^y(1-pq)^{n-y}$$

$$= \begin{cases} \boldsymbol{\binom{n}{y} p^y q^y (1-pq)^{n-y}} & \textbf{if } \boldsymbol{0 \le y \le n} \\ \boldsymbol{0} & \textbf{otherwise} \end{cases}$$

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   e. Find the conditional PMF $p_{X|Y}(x|y)$. Do this algebraically using the preceding answers.

   Solution:
   Using the solutions to parts (c) and (d), we get

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

$$= \frac{\boldsymbol{\binom{n}{x}\binom{x}{y} p^x q^y (1-p)^{n-x}(1-q)^{x-y}}}{\boldsymbol{\binom{n}{y} p^y q^y (1-pq)^{n-y}}}$$

   Optionally, we can simplify slightly by using the identity $\binom{n}{x}\binom{x}{y} = \binom{n}{y}\binom{n-y}{x-y}$, then cancel:

$$p_{X|Y}(x|y) = \begin{cases} \boldsymbol{\binom{n-y}{x-y} p^{x-y} \frac{(1-p)^{n-x}(1-q)^{x-y}}{(1-pq)^{n-y}}} & \textbf{if } \boldsymbol{0 \le y \le x \le n} \\ \boldsymbol{0} & \textbf{otherwise} \end{cases}$$

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Section 3.2: Cumulative Distribution Functions

2. Find the PDF, the mean, and the variance of the random variable X with CDF

$$F_X(x) = \begin{cases} 1 - \dfrac{a^3}{x^3} & \text{if } x \geq a \\ 0 & \text{if } x < a \end{cases}$$

where a is a positive constant.

Solution:

$$p_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} \boldsymbol{\dfrac{3a^3}{x^4}} & \textbf{if } \boldsymbol{x \geq a} \\ \mathbf{0} & \textbf{if } \boldsymbol{x < a} \end{cases}$$

$$E[X] = \int_a^\infty x \cdot \frac{3a^3}{x^4}\, dx$$

$$= 3a^3 \int_a^\infty x^{-3}\, dx$$

$$= -\frac{3}{2} a^3 x^{-2} \Big|_a^\infty$$

$$= -\frac{3}{2} a^3 (0 - a^{-2})$$

$$= \boldsymbol{\frac{3a}{2}}$$

$$E[X^2] = \int_a^\infty x^2 \cdot \frac{3a^3}{x^4}\, dx$$

$$= 3a^3 \int_a^\infty x^{-2}\, dx$$

$$= -3a^3 x^{-1} |_a^\infty$$

$$= -3a^3 (0 - a^{-1})$$

$$= 3a^2$$

$$Var(X) = E[X^2] - E[X]^2 = 3a^2 - \left(\frac{3a}{2}\right)^2 = \left(3 - \frac{9}{4}\right) a^2 = \boldsymbol{\frac{3a^2}{4}}$$

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3. You are allowed to take a certain test three times, and your final score will be the maximum of the test scores. Your score in test i, where $i = 1, 2, 3$, takes one of the values from i to 10 with equal probability $1/(11 - i)$, independently of the scores in other tests. What is the PMF of the final score?

Solution:
Let X_i be your score on test i, for $1 \le i \le 3$, and let $X = \max(X_1, X_2, X_3)$. We are given the PMFs

$$p_{X_1}(k) = \begin{cases} \frac{1}{10} & \text{if } 1 \le k \le 10 \\ 0 & \text{otherwise} \end{cases}, \quad p_{X_2}(k) = \begin{cases} \frac{1}{9} & \text{if } 2 \le k \le 10 \\ 0 & \text{otherwise} \end{cases}, \quad p_{X_3}(k) = \begin{cases} \frac{1}{8} & \text{if } 3 \le k \le 10 \\ 0 & \text{otherwise} \end{cases}.$$

The CDFs are therefore

$$F_{X_1}(x) = \begin{cases} 0 & x < 1 \\ \frac{\lfloor x \rfloor}{10} & 1 \le x < 10 \\ 1 & x \ge 10 \end{cases}, \; F_{X_2}(x) = \begin{cases} 0 & x < 2 \\ \frac{\lfloor x - 1 \rfloor}{9} & 2 \le x < 10 \\ 1 & x \ge 10 \end{cases}, \; F_{X_3}(x) = \begin{cases} 0 & x < 3 \\ \frac{\lfloor x - 2 \rfloor}{8} & 3 \le x < 10 \\ 1 & x \ge 10 \end{cases}.$$

Using independence, we have

$$\begin{aligned} F_X(x) &= P(X \le x) \\ &= P(X_1 \le x, X_2 \le x, X_3 \le x) \\ &= P(X_1 \le x)P(X_2 \le x)P(X_3 \le x) \\ &= F_{X_1}(x)F_{X_2}(x)F_{X_3}(x) \\ &= \begin{cases} \mathbf{0} & \boldsymbol{x < 3} \\ \frac{\lfloor \boldsymbol{x} \rfloor}{\mathbf{10}} \cdot \frac{\lfloor \boldsymbol{x} - \mathbf{1} \rfloor}{\mathbf{9}} \cdot \frac{\lfloor \boldsymbol{x} - \mathbf{2} \rfloor}{\mathbf{8}} & \boldsymbol{3 \le x < 10} \\ \mathbf{1} & \boldsymbol{x \ge 10} \end{cases} \end{aligned}$$

Thus

$$\begin{aligned} p_X(k) &= \boldsymbol{F_X(k) - F_X(k-1)} \\ &= \begin{cases} \frac{\mathbf{1}}{\mathbf{120}} & \boldsymbol{k = 3} \\ \frac{\boldsymbol{(k-1)(k-2)}}{\mathbf{240}} & \boldsymbol{k = 4, 5, 6, 7, 8, 9} \\ \frac{\mathbf{3}}{\mathbf{10}} & \boldsymbol{k = 10} \\ \mathbf{0} & \textbf{Otherwise} \end{cases} \end{aligned}$$

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4. The median of a random variable X is a number m that satisfies $F_X(m) = 1/2$. Find the median of the exponential random variable with parameter λ.

Solution:
We are given that

$$\frac{1}{2} = F_X(m) = \int_{-\infty}^{m} f_X(x)dx = \int_{0}^{m} \lambda e^{-\lambda x}\, dx = \left. -e^{-\lambda x} \right|_0^m = 1 - e^{-\lambda m}$$

thus

$$e^{-\lambda m} = 1 - \frac{1}{2} = \frac{1}{2}$$

$$-\lambda m = \ln\left(\frac{1}{2}\right) = -\ln(2)$$

$$m = \frac{\mathbf{ln(2)}}{\boldsymbol{\lambda}}$$

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Section 3.3: Normal Random Variables

5. A radar tends to overestimate the distance of an aircraft, and the error is a normal random variable with a mean of 50 meters and a standard deviation 100 meters. What is the probability that the measured distance will be smaller than the true distance?

Solution:
Let X be the radar error. We are given that X is normal with $\mu = 50$ and $\sigma = 100$, and we seek the probability that X is negative, i.e. $P(X < 0)$. Thus

$$P(X < 0) = P\left(\frac{X - 50}{100} < \frac{0 - 50}{100}\right) = P(Y < -0.5)$$

where $Y = \frac{X-50}{100}$ is a standard normal random variable. By consulting the normal table, we have

$$P(X < 0) = \Phi(-0.5) = 1 - \Phi(0.5) = 1 - (.6915) = \mathbf{.3085}$$

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