**CSE 107**
**Homework Assignment 4**
**Solutions**

Section 2.6: Conditioning

1. This is an extension of problem 4 from hw3, and utilizes the answers to parts a, b and c from that problem.

   Alvin shops for probability books for $K$ hours, where $K$ is a random variable that is equally likely to be 1, 2, 3, or 4. The number of books $N$ that he buys is random and depends on how long he shops according to the conditional PMF

$$p_{N|K}(n|k) = \begin{cases} \frac{1}{k} & \text{if } n = 1, \dots, k \\ 0 & \text{otherwise} \end{cases}$$

   d) Find the conditional mean and variance of $K$, given that he bought at least 2 but no more than 3 books.
   **Solution:**
   Let $A = \{2 \leq N \leq 3\} = \{N = 2\} \cup \{N = 3\}$. We first determine the conditional PMF

$$p_{K|A}(k) = \frac{P(\{K = k\} \cap A)}{P(A)}$$

$$= \frac{P(K = k, N = 2) + P(K = k, N = 3)}{P(A)}$$

$$= \frac{p_{N,K}(2, k) + p_{N,K}(3, k)}{P(A)}$$

   Using the solution to parts (a) and (b), we have

$$p_{N,K}(2, k) + P_{N,K}(3, k) = \begin{cases} \frac{1}{8} \\ \frac{1}{12} + \frac{1}{12} \\ \frac{1}{16} + \frac{1}{16} \end{cases} = \begin{cases} \frac{1}{8} & \text{if } k = 2 \\ \frac{1}{6} & k = 3 \\ \frac{1}{8} & k = 4 \end{cases}$$

   and

$$P(A) = p_N(2) + p_N(3) = \frac{13}{48} + \frac{7}{48} = \frac{20}{48} = \frac{5}{12}$$

   hence

$$p_{K|A}(k) = \begin{cases} \frac{3}{10} & \text{if } k = 2 \\ \frac{2}{5} & k = 3 \\ \frac{3}{10} & k = 4 \end{cases}$$

Thus

$$E[K|A] = 2 \cdot \left(\frac{3}{10}\right) + 3 \cdot \left(\frac{2}{5}\right) + 4 \cdot \left(\frac{3}{10}\right) = \mathbf{3}$$

$$E[K^2|A] = 2^2 \cdot \left(\frac{3}{10}\right) + 3^2 \cdot \left(\frac{2}{5}\right) + 4^2 \cdot \left(\frac{3}{10}\right)$$

$$= \frac{6 + 18 + 24}{5} = \frac{48}{5}$$

and therefore

$$Var(K|A) = E[K^2|A] - E[K|A]^2$$

$$= \frac{48}{5} - 3^2$$

$$= \frac{48 - 45}{5} = \mathbf{\frac{3}{5}}$$

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e) The cost of each book is a random variable with mean \$30. What is the expected value of his total expenditure? Hint: Condition on the events $\{N = 1\}$, $\{N = 2\}$, $\{N = 3\}$ and $\{N = 4\}$ and use the total expectation theorem.

**Solution:**
Let $X$ be the total expenditure on books. The total expectation theorem gives

$$E[X] = E[X|N = 1]p_N(1) + E[X|N = 2]p_N(2) + E[X|N = 3]p_N(3) + E[X|N = 4]p_N(4)$$

Let $Y$ be the cost of a single book, and observe that $X = NY$. We are given that $E[Y] = 30$, hence

$$E[X|N = 1] = E[NY|N = 1] = E[Y] = 30,$$

$$E[X|N = 2] = E[NY|N = 2] = E[2Y] = 2 \cdot 30 = 60,$$

$$E[X|N = 3] = E[NY|N = 3] = E[3Y] = 3 \cdot 30 = 90,$$

and

$$E[X|N = 4] = E[NY|N = 4] = E[4Y] = 4 \cdot 30 = 120.$$

Using the solution to 4b from hw3 which gives us the PMF $p_N(n)$, we have

$$E[X] = 30 \cdot \frac{25}{48} + 60 \cdot \frac{13}{48} + 90 \cdot \frac{7}{48} + 120 \cdot \frac{3}{48} = \frac{2520}{48} = \mathbf{\frac{105}{2}} = \mathbf{52.5}$$

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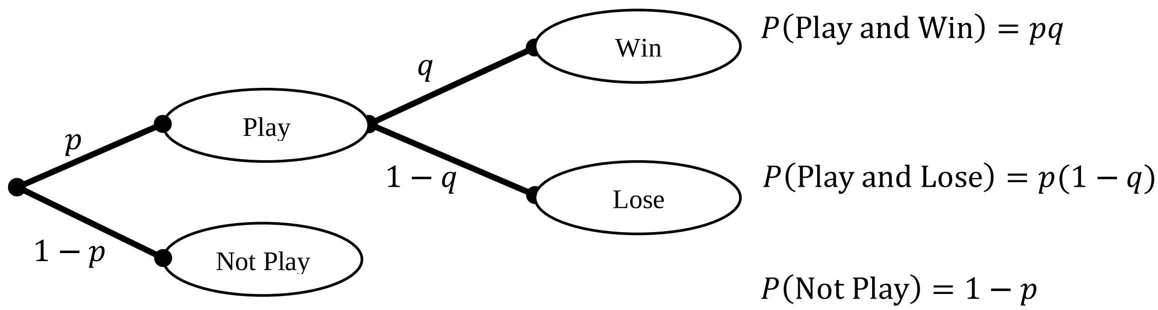
Section 2.7: Independence

2. Joe Lucky plays the lottery on any given week with probability $p$, independently of whether he played on any other week. Each time he plays, he has a probability $q$ of winning, again independently of everything else. During a fixed time period of $n$ weeks, let $X$ be the number of weeks that he played the lottery and $Y$ be the number of weeks that he won.

   a. What is the probability that he played the lottery on any particular week, given that he did not win on that week?

   **Solution:**
   In any given week, the following tree diagram applies.




   Therefore $P(\text{Win}) = P(\text{Play and Win}) = pq$, $P(\text{Not Win}) = 1 - pq$ and

$$P(\text{Play} \mid \text{Not Win}) = \frac{P(\text{Play and Not Win})}{P(\text{Not Win})} = \boldsymbol{\frac{p(1-q)}{1-pq}}$$

   **Alternate Solution:**
   Define the events $A = \{\text{Play}\}$ and $B = \{\text{Win}\}$. Then by Bayes theorem,

$$P(A|B^c) = \frac{P(B^c|A) \cdot P(A)}{P(B^c|A) \cdot P(A) + P(B^c|A^c) \cdot P(A^c)}$$

$$= \frac{(1-q)p}{(1-q)p + 1 \cdot (1-p)}$$

$$= \frac{p(1-q)}{p - pq + 1 - p}$$

$$= \boldsymbol{\frac{p(1-q)}{1-pq}}$$

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b. Find the conditional PMF $p_{Y|X}(y|x)$.

**Solution:**
Given that Joe plays on $x$ days, $Y$ is binomial with parameters $x$ and $q$. Thus

$$p_{Y|X}(y|x) = \begin{cases} \boldsymbol{\binom{x}{y} q^y (1-q)^{x-y}} & \textbf{if } \boldsymbol{0 \le y \le x} \\ \mathbf{0} & \textbf{otherwise} \end{cases}$$

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c. Find the joint PMF $p_{X,Y}(x,y)$.

**Solution:**
The number of weeks in which Joe plays is binomial random variable with parameters $n$ and $p$, so that

$$p_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & \text{if } 0 \le x \le n \\ 0 & \text{otherwise} \end{cases}.$$

Combining this with the answer to part (b) gives

$$\begin{aligned} p_{X,Y}(x,y) &= p_{Y|X}(y|x) \cdot p_X(x) \\ &= \binom{x}{y} q^y (1-q)^{x-y} \cdot \binom{n}{x} p^x (1-p)^{n-x} \end{aligned}$$

hence

$$p_{X,Y}(x,y) = \begin{cases} \boldsymbol{\binom{n}{x}\binom{x}{y} p^x q^y (1-p)^{n-x} (1-q)^{x-y}} & \textbf{if } \boldsymbol{0 \le y \le x \le n} \\ \mathbf{0} & \textbf{otherwise} \end{cases}$$

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Section 3.1: Continuous Random Variables and PDFs

3. The runner-up in a road race is given a reward that depends on the difference between his time and the winner's time. He is given 10 dollars for being one minute behind, 6 dollars for being one to three minutes behind, 2 dollars for being 3 to 6 minutes behind, and nothing otherwise. Given that the difference between his time and the winner's time is uniformly distributed between 0 and 12 minutes, find the mean and variance of the reward of the runner-up.

**Solution:**
Let *D* be the difference in time between the winner and the runner-up. We are given that *D* is uniform on $[0, 12]$, so that its PDF is

$$f_D(x) = \begin{cases} \frac{1}{12} & \text{if } 0 \le x \le 12 \\ 0 & \text{otherwise} \end{cases}$$

Thus $P(0 \leq D \leq 1) = 1/12$, $P(1 < D \leq 3) = 2/12$ and $P(3 < D \leq 6) = 3/12$. Now let $X$ be the reward of the runner up. Then $X$ is a discrete random variable with PMF

$$p_X(k) = \begin{cases} \frac{1}{12} & \text{if } k = 10 \\ \frac{2}{12} & k = 6 \\ \frac{3}{12} & k = 2 \\ \frac{6}{12} & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Therefore

$$E[X] = 10 \cdot \frac{1}{12} + 6 \cdot \frac{2}{12} + 2 \cdot \frac{3}{12} = \frac{28}{12} = \mathbf{\frac{7}{3}} = \$\mathbf{2.33}$$

$$E[X^2] = 10^2 \cdot \frac{1}{12} + 6^2 \cdot \frac{2}{12} + 2^2 \cdot \frac{3}{12} = \frac{184}{12} = \frac{46}{3}$$

and

$$Var(X) = E[X^2] - E[X]^2 = \frac{46}{3} - \left(\frac{7}{3}\right)^2 = \frac{138 - 49}{9} = \mathbf{\frac{89}{9}} = \mathbf{9.8889}$$

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4. Let X be a random variable with PDF

$$f_X(x) = \begin{cases} \frac{2x}{3} & \text{if } 1 < x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

and let $Y = X^2$. Calculate $E[Y]$ and $Var(Y)$.

**Solution:**

$$E[Y] = E[X^2] = \int_1^2 x^2 \cdot \frac{2}{3} x \, dx = \frac{2}{3} \int_1^2 x^3 \, dx = \frac{2}{3} \cdot \frac{1}{4} x^4 \Big|_1^2 = \frac{1}{6}(2^4 - 1) = \mathbf{\frac{5}{2}}$$

$$E[Y^2] = E[X^4] = \int_1^2 x^4 \cdot \frac{2}{3} x \, dx = \frac{2}{3} \int_1^2 x^5 \, dx = \frac{2}{3} \cdot \frac{1}{6} x^6 \Big|_1^2 = \frac{1}{9}(2^6 - 1) = 7$$

$$Var(Y) = E[Y^2] - E[Y]^2 = 7 - \left(\frac{5}{2}\right)^2 = \frac{28 - 25}{4} = \mathbf{\frac{3}{4}}$$

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