CSE 107
Homework Assignment 3
Solutions

<u>Section 2.4</u>: Expectation, Mean and Variance

1. A class with 300 students consists of 250 undergraduate students and 50 graduate students. The probability that an undergraduate student gets an A is $1/3$, and the probability that a graduate student gets an A is $1/2$. Let X be the number of students who get an A. Determine the expectation $E[X]$.

   Hint: define random variables

$$U_i = \begin{cases} 1 & \text{if undergraduate student } i \text{ gets an A} \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } 1 \le i \le 250$$

   and

$$G_j = \begin{cases} 1 & \text{if graduate student } j \text{ gets an A} \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } 1 \le j \le 50$$

   Write X in terms of the U_i and G_j, then use linearity of expectation.

   Solution:
   Note that U_i and G_j from the hint are binomial with parameters $1/3$ and $1/2$, respectively. Therefore $E[U_i] = 1/3$ and $E[G_j] = 1/2$. Since

$$X = \sum_{i=1}^{250} U_i \; + \; \sum_{j=1}^{50} G_j$$

   we have that

$$E[X] = 250 \cdot \frac{1}{3} + 50 \cdot \frac{1}{2} = \frac{\mathbf{325}}{\mathbf{3}} = \mathbf{108.334}$$

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<u>Section 2.5</u>: Joint PMFs of multiple random variables

2. The UCSC football team wins any one game with probability p, and loses it with probability $1 - p$. Its performance in each game is independent of its performance in other games. Let L_1 be the number of losses before its first win, and let L_2 be the number of losses after its first win and before its second win. Find the joint PMF of L_1 and L_2.

   Solution:
   Observe that $p_{L_1,L_2}(i,j) = P(L_1 = i, L_2 = j)$ is the probability a sequence of i losses, followed by 1 win, followed by j losses, followed by 1 win. Therefore

$$\boldsymbol{p_{L_1,L_2}(i,j) = (1-p)^i \cdot p \cdot (1-p)^j \cdot p = (1-p)^{i+j} \cdot p^2}$$

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3. A class of n students takes a test in which each student gets an A with probability p, a B with probability q, and a grade below B with probability $1-p-q$, independently of any other student. Let X and Y be the numbers of students that getting A's and a B's, respectively. Determine the joint PMF $p_{X,Y}(x,y)$.

Solution:
Let x and y be integers satisfying $x \geq 0$, $y \geq 0$ and $x+y \leq n$. For simplicity, assume that any student who does not get an A or B gets a C. Then an assignment of grades in which the number of A's is x, and the number of B's is y, can be represented as a string over the alphabet $\{A, B, C\}$ of length n containing exactly x A's, exactly y B's, and hence $(n-x-y)$ Cs. The probability of one such string (i.e. grade assignment) is

$$p^x \cdot q^y \cdot (1-p-q)^{n-x-y}.$$

The number of such strings is the multinomial coefficient

$$\binom{n}{x, y, (n-x-y)} = \frac{n!}{x!\,y!\,(n-x-y)!}.$$

(See section 1.6 of the text, pp. 49-51 and related examples.) Therefore

$$\boldsymbol{p_{X,Y}(x,y) = \begin{cases} \binom{n}{x, y, (n-x-y)} \cdot p^x \cdot q^y \cdot (1-p-q)^{n-x-y} & \text{for } x \geq 0, y \geq 0, x+y \leq n \\ 0 & \text{otherwise} \end{cases}}$$

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Section 2.6: Conditioning

4. Alvin shops for probability books for K hours, where K is a random variable that is equally likely to be 1, 2, 3, or 4. The number of books N that he buys is random and depends on how long he shops according to the conditional PMF

$$p_{N|K}(n|k) = \begin{cases} \frac{1}{k} & \text{if } n = 1, \dots, k \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the joint PMF of K and N.

Solution:
We are given that

$$p_K(k) = \begin{cases} \frac{1}{4} & \text{if } k = 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$$

and

$$p_{N|K}(n|k) = \begin{cases} 1 & \text{if } n = 1 & \text{and } k = 1 \\ \frac{1}{2} & n = 1, 2 & k = 2 \\ \frac{1}{3} & n = 1, 2, 3 & k = 3 \\ \frac{1}{4} & n = 1, 2, 3, 4 & k = 4 \\ 0 & \text{otherwise} \end{cases}$$

Therefore

$$p_{N,K}(n,k) = p_K(k) \cdot p_{N|K}(n|k)$$

$$= \begin{cases} \mathbf{\frac{1}{4}} & \textbf{if } \boldsymbol{n = 1} & \textbf{and } \boldsymbol{k = 1} \\ \mathbf{\frac{1}{8}} & \boldsymbol{n = 1, 2} & \boldsymbol{k = 2} \\ \mathbf{\frac{1}{12}} & \boldsymbol{n = 1, 2, 3} & \boldsymbol{k = 3} \\ \mathbf{\frac{1}{16}} & \boldsymbol{n = 1, 2, 3, 4} & \boldsymbol{k = 4} \\ \mathbf{0} & \textbf{otherwise} \end{cases}$$

(b) Find the marginal PMF of N.

Solution:

$$p_N(n) = \sum_{k=1}^{4} P_{N,K}(n,k) = \begin{cases} \frac{1}{4} + \frac{1}{8} + \frac{1}{12} + \frac{1}{16} & \text{if } n = 1 \\ \frac{1}{8} + \frac{1}{12} + \frac{1}{16} & n = 2 \\ \frac{1}{12} + \frac{1}{16} & n = 3 \\ \frac{1}{16} & n = 4 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \mathbf{\frac{25}{48}} & \textbf{if } \boldsymbol{n = 1} \\ \mathbf{\frac{13}{48}} & \boldsymbol{n = 2} \\ \mathbf{\frac{7}{48}} & \boldsymbol{n = 3} \\ \mathbf{\frac{1}{16}} & \boldsymbol{n = 4} \\ \mathbf{0} & \textbf{otherwise} \end{cases}$$

(c) Find the conditional PMF of K given that $N = 2$.

Solution:

$$p_{K|N}(k|2) = \frac{p_{N,K}(2,k)}{p_N(2)} = \frac{p_{N,K}(2,k)}{13/48}$$

$$= \begin{cases} \dfrac{1/8}{13/48} & \text{if } k = 2 \\ \dfrac{1/12}{13/48} & k = 3 \\ \dfrac{1/16}{13/48} & k = 4 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \boldsymbol{\dfrac{6}{13}} & \textbf{if } \boldsymbol{k = 2} \\ \boldsymbol{\dfrac{4}{13}} & \boldsymbol{k = 3} \\ \boldsymbol{\dfrac{3}{13}} & \boldsymbol{k = 4} \\ \mathbf{0} & \textbf{otherwise} \end{cases}$$

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