CSE 107
Homework Assignment 1
Solutions to Selected Problems

Section 1.2:

1. We roll a four-sided die once and then we roll it as many times as is necessary to obtain a different face than the one obtained in the first roll. Let the outcome of the experiment be (r_1, r_2) where r_1 and r_2 are the results of the first and the last rolls, respectively. Assume that all possible outcomes have equal probability. Find the probability that:

   (a) r_1 is even.
   (b) Both r_1 and r_2 are even.
   (c) $r_1 + r_2 < 5$.

   Solution:
   The sample space can be visualized as

X	12	13	14
21	X	23	24
31	32	X	34
41	42	43	X

   We have $|\Omega| = 12$, and since each outcome is equally likely, each has probability $1/12$.

   (a) r_1 is even. $\boldsymbol{P(\{21, 23, 24, 41, 42, 43\}) = 6/12 = 1/2}$

   (b) Both r_1 and r_2 are even. $\boldsymbol{P(\{24,\ 42\}) = 2/12 = 1/6}$

   (c) $r_1 + r_2 < 5$. $\boldsymbol{P(\{12, 21, 13, 31\}) = 4/12 = 1/3}$

2. You enter a special kind of chess tournament, whereby you play one game with each of three opponents, but you get to choose the order in which you play your opponents. You win the tournament if you win two games in a row. You know your probability of a win against each of the three opponents. What is your probability of winning the tournament, assuming that you choose the optimal order of playing the opponents?

   Solution:
   Label the players 1, 2 and 3 by increasing strength, so that player 1 is the weakest and player 3 the strongest. Let p_i be your probability of winning against player i, so that $p_1 > p_2 > p_3$. It will be necessary to determine the optimal order of playing the three opponents. Suppose for instance that the order is 123, and define events $A = \{\text{you win the first two}\}$, $B = \{\text{you win the last two}\}$. Then $A \cap B = \{\text{you win all three}\}$. If we assume the matches are independent, we have $P(A) = p_1 p_2$, $P(B) = p_2 p_3$ and $P(A \cap B) = p_1 p_2 p_3$. The probability of winning the tournament in this case is then $P(A \cup B) = P(A) + P(B) - P(A \cap B) = p_1 p_2 + p_2 p_3 - p_1 p_2 p_3$.

Doing this same calculation for each of the 6 possible orders gives the following results.

Order	Probability of winning 2 in a row	
123	$p_1p_2 + p_2p_3 - p_1p_2p_3$	
132	$p_1p_3 + p_2p_3 - p_1p_2p_3$	
213	$\boldsymbol{p_1p_2 + p_1p_3 - p_1p_2p_3}$	**maximum**
231	$p_1p_3 + p_2p_3 - p_1p_2p_3$	
312	$\boldsymbol{p_1p_2 + p_1p_3 - p_1p_2p_3}$	**maximum**
321	$p_1p_2 + p_2p_3 - p_1p_2p_3$	

We must determine which of these probabilities is the greatest. From $p_1 > p_2 > p_3$, it follows that $p_1p_2 > p_1p_3 > p_2p_3$, and further that $p_1p_2 + p_1p_3 > p_1p_2 + p_2p_3 > p_1p_3 + p_2p_3$. Therefore the maximum probability is $\boldsymbol{p_1p_2 + p_1p_3 - p_1p_2p_3}$, which is achieved by putting the weakest player in the middle.

Section 1.3:

4. We roll two fair 6-sided dice. Each one of the 36 possible outcomes is assumed to be equally likely.

   (a) Find the probability that doubles were rolled.
   (b) Given that the roll resulted in a sum of 4 or less, find the conditional probability that doubles were rolled.
   (c) Find the probability that at least one die is a 6.
   (d) Given that the two dice land on different numbers, find the conditional probability that at least one die is a 6.

   Solutions:
   (a) $P(\text{doubles}) = 6/36 = \mathbf{1/6}$
   (b) $P(\text{doubles}|\text{sum} \leq 4) = 2/6 = \mathbf{1/3}$
   (c) $P(\text{at least one } 6) = \mathbf{11/36}$
   (d) $P(\text{at least one } 6 \mid \text{different}) = 10/30 = \mathbf{1/3}$

Section 1.4:

5. Alice and Bob have $2n + 1$ coins, each with probability of a head equal to $1/2$. Bob tosses $n + 1$ coins, while Alice tosses the remaining n coins. Show that the probability that after all the coins have been tossed, Bob will have gotten more heads than Alice is $1/2$.

   Solution:
   Let E be the event that, after all coins are tossed, Bob has more heads. Denote by a_n the number of heads Alice tosses after n flips, and by b_n the number of heads Bob tosses after n flips. Consider the events $\{a_n < b_n\}$, $\{a_n > b_n\}$ and $\{a_n = b_n\}$. Since, in these events, Alice and Bob flip the same number of identical coins, symmetry implies that $P(a_n < b_n) = P(a_n > b_n)$. Also, since

$$\Omega = \{a_n < b_n\} \cup \{a_n > b_n\} \cup \{a_n = b_n\}$$

is a disjoint union, we also have $P(a_n < b_n) + P(a_n > b_n) + P(a_n = b_n) = 1$. If Bob is ahead after n flips, he will certainly be ahead after his extra flip, and therefore

$$P(E|\{a_n < b_n\}) = 1.$$

Similarly, if Alice is ahead after n flips, Bob's extra flip will not put him ahead, and hence

$$P(E|\{a_n > b_n\}) = 0.$$

Finally, if Alice and Bob are tied after n flips, Bob's extra flip puts him ahead if and only if he tosses a head, and therefore

$$P(E|\{a_n = b_n\}) = 1/2.$$

The total probability theorem now gives

$$\begin{aligned} P(E) &= 1 \cdot P(a_n < b_n) + 0 \cdot P(a_n > b_n) + (1/2) \cdot P(a_n = b_n) \\ &= \frac{1}{2} \cdot 2 \cdot P(a_n < b_n) + \frac{1}{2} \cdot P(a_n = b_n) \\ &= \frac{1}{2}[2 \cdot P(a_n < b_n) + P(a_n = b_n)] \\ &= \frac{1}{2}[P(a_n < b_n) + P(a_n > b_n) + P(a_n = b_n)] \\ &= \frac{1}{2} \end{aligned}$$

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Remarks on Lab Assignment 1:
If all the fair coins in this problem are replaced by weighted coins with probability of heads equal to p, the only change to the above analysis is that now $P(E|\{a_n = b_n\}) = p$. We then arrive, in a similar manner to that above, at the expression

$$P(E) = \frac{1}{2} \cdot P(a_n \neq b_n) + p \cdot P(a_n = b_n).$$

Why does this probability approach $1/2$ as n becomes large? It does so because there are many more ways in which event $\{a_n \neq b_n\}$ can occur ($(n^2 - n)$ ways to be exact) than $\{a_n = b_n\}$ can occur (n ways). Thus, in the limit as $n \to \infty$ we have $P(a_n \neq b_n) \to 1$ and $P(a_n = b_n) \to 0$. ■

6. A magnetic tape storing information in binary form has been corrupted, so it can only be read with bit errors. The probability that you correctly detect a 0 is 0.9, while the probability that you correctly detect a 1 is 0.85. Each digit is a 1 or a 0 with equal probability. Given that you read a 1, what is the probability that this is a correct reading?

Solution:
Define the events $T_i = \{\text{bit } i \text{ was transmitted}\}$ and $D_i = \{\text{bit } i \text{ was detected}\}$, for $i = 0, 1$. We are given that

$$P(D_0|T_0) = .9 \text{ and hence } P(D_1|T_0) = .1,$$

$$P(D_1|T_1) = .85,$$

and

$$P(T_0) = P(T_1) = .5.$$

We must determine $P(T_1|D_1)$. Using Baye's rule we have

$$P(T_1|D_1) = \frac{P(D_1|T_1) \cdot P(T_1)}{P(D_1)}$$

$$= \frac{P(D_1|T_1) \cdot P(T_1)}{P(D_1|T_1) \cdot P(T_1) + P(D_1|T_0) \cdot P(T_0)}$$

$$= \frac{(.85)(.5)}{(.85)(.5) + (.1)(.5)}$$

$$= \mathbf{.8947}$$

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Section 1.5:

7. A peculiar six-sided die has uneven faces. In particular, the faces showing 1 or 6 are 1×1.5 inches, the faces showing 2 or 5 are 1×0.4 inches, and the faces showing 3 or 4 are 0.4×1.5 inches. Assume that the probability of a particular face coming up is proportional to its area. We independently roll the die twice. What is the probability that we get doubles?

Solution:
Calculating the sum of the areas of each face we have area $= 2(1.5 + .4 + .6) = 5$. Dividing the area of each face by the constant 5 gives the following table of probabilities.

Face	1	2	3	4	5	6
Probability	.3	.08	.12	.12	.08	.3

Let $D = \{11, 22, 33, 44, 55, 66\}$ be the event that the outcome is doubles. Then

$$P(D) = (.3)^2 + (.08)^2 + (.12)^2 + (.12)^2 + (.08)^2 + (.3)^2$$

$$= \mathbf{.2216}$$

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8. A parking lot consists of a single row containing n parking spaces ($n \geq 2$). Mary arrives when all spaces are free. Tom is the next person to arrive. Each person makes an equally likely choice among all available spaces at the time of arrival. Describe the sample space. Obtain $P(A)$, the probability the parking spaces selected by Mary and Tom are at most 2 spaces apart.

Solution:
Denote by A the event that Mary and Tom are at most 2 spaces apart. Let E be the event that Mary parks in one of the 2 *end* spaces. Then $P(E) = 2/n$ and $P(A|E) = 2/(n-1)$, since of the $n-1$ remaining spaces, only 2 are within 2 of Mary. Let S be the event that Mary parks in one of the 2 spaces *second* from the end. In this case $P(S) = 2/n$ and $P(A|S) = 3/(n-1)$. Finally, let M be the event that Mary parks in one of the *middle* $n-4$ spaces. Then $P(M) = (n-4)/n$ and $P(A|M) = 4/(n-1)$. By the total probability theorem

$$P(A) = P(E) \cdot P(A|E) + P(S) \cdot P(A|S) + P(M) \cdot P(A|M)$$

$$= \frac{2}{n} \cdot \frac{2}{n-1} + \frac{2}{n} \cdot \frac{3}{n-1} + \frac{n-4}{n} \cdot \frac{4}{n-1}$$

$$= \frac{4 + 6 + 4n - 16}{n(n-1)}$$

$$= \frac{\mathbf{(4n-6)}}{\mathbf{n(n-1)}}.$$

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9. A particular jury consists of 7 jurors. Each juror has a 0.2 chance of making the wrong decision, independently of the others. If the jury reaches a decision by majority rule, what is the probability that it will reach a wrong decision?

Solution:
The number of wrong jurors amongst the 7 is given by a Binomial probability:

$$P(\#\text{wrong} = k) = \binom{7}{k} (.2)^k (.8)^{7-k}$$

Therefore, the probability that a majority of jurors are wrong is

$$P(\#\text{wrong} \geq 4) = \sum_{k=4}^{7} \binom{7}{k} (.2)^k (.8)^{7-k} = \mathbf{.0333}.$$

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