cse 107 9-29-22

- supplemental lecture! Yuja

1.5 Independence

what if knowledge that B occurs tells us nothing as to A.

* $P(A|B) = P(A)$

we say A is independent of B.

$$\frac{P(A\cap B)}{P(B)} = P(A)$$

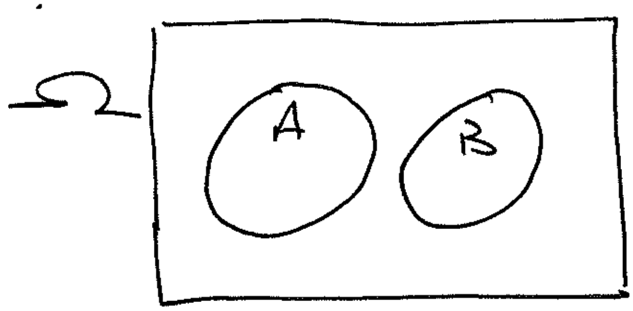
** $P(A\cap B) = P(A)\cdot P(B)$

Defn

We say $A, B \subseteq \Omega$ are **independent** iff

$$P(A \cap B) = P(A) \cdot P(B)$$

note:



$A \cap B = \emptyset$

$\Rightarrow P(A \cap B) = 0$

disjoint $\neq$ independent

so if $P(A) \neq 0$, $P(B) \neq 0$

$$P(A \cap B) \neq P(A)P(B)$$

Ex. Throw 2 fair dice.

Let

$$A_i = \{1^{st} \text{ die is } i\} \quad (1 \leq i \leq 6)$$

$$B_j = \{2^{nd} \text{ die is } j\} \quad (1 \leq j \leq 6)$$

If A_i, B_j are independent, then

$$P((i,j)) = P(A_i \cap B_j) = P(A_i) \cdot P(B_j) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}.$$

note: discrete uniform $\Leftrightarrow$ independence

Assume, unless otherwise stated, dice are independent.

Let $C = \{\text{sum} = 5\}$

$= \{41, 32, 23, 14\}$

$\therefore P(C) = \frac{4}{36} = \frac{1}{9}$

Are A_2 and C independent?

$P(A_2) = \frac{6}{36} = \frac{1}{6}$

$A_2 \cap C = \{23\}$

$P(A_2 \cap C) = \frac{1}{36}$

$P(A_2) \cdot P(C) = \frac{1}{6} \cdot \frac{1}{9} = \frac{1}{54}$

answer: $\boxed{\text{no}}$

Let $D = \{\text{doubles}\}$

$= \{11, 22, 33, 44, 55, 66\}$

Are A_2 and D independent?

$$P(D) = \frac{6}{36} = \frac{1}{6}$$

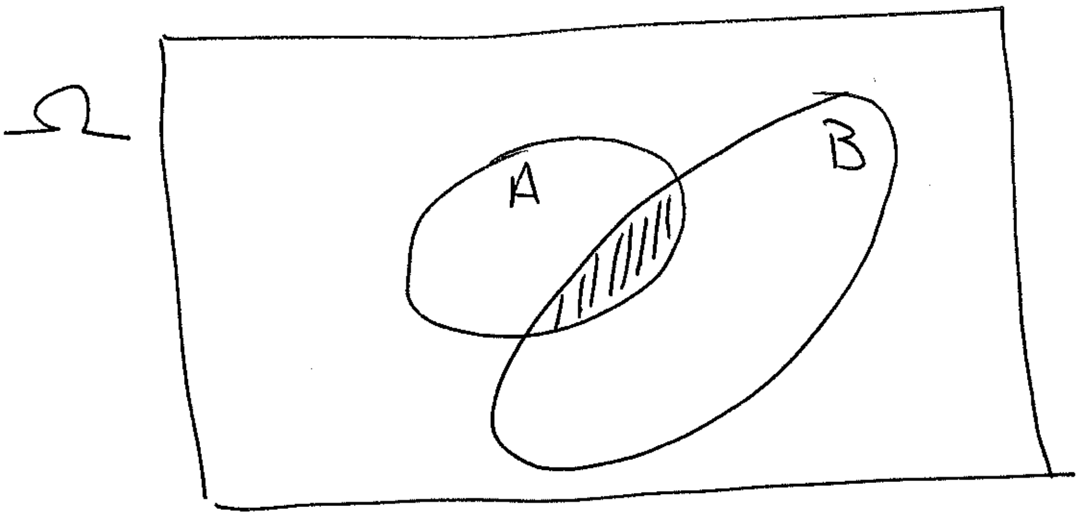
$$A_2 \cap D = \{22\}$$

$$\therefore \; P(A_2 \cap D) = \frac{1}{36}$$

$$P(A_2) \cdot P(D) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

answer: $\boxed{\text{yes}}$

Visualizing Independence!



think of Probability as (normalized) area.

the Proportion of $A \cap B$ to B
equals Proportion of A to Ω

$$\frac{\text{area}(A \cap B)}{\text{area}(B)} = \frac{\text{area}(A)}{\text{area}(\Omega)}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A)$$

Conditional Independence

Recall: if $P(C) > 0$, then

$$P(\cdot \mid C)$$

we can therefore define independence w.r.t. this new probability law.

Defn

we say $A, B \subseteq \Omega$ are *conditionally independent given C* iff

$$\boxed{P(A \cap B \mid C) = P(A \mid C) \cdot P(B \mid C)}$$

equivalent definition:

$$P(A \cap B \mid C) = \frac{P(A \cap B \cap C)}{P(C)}$$

$$= \frac{\cancel{P(C)} \cdot P(B \mid C) \cdot P(A \mid B \cap C)}{\cancel{P(C)}}$$

$$= P(B \mid C) \cdot P(A \mid B \cap C)$$

$$\therefore \cancel{P(B \mid C)} \cdot P(A \mid B \cap C) = P(A \mid C) \cdot \cancel{P(B \mid C)}$$

$$\therefore \boxed{P(A \mid B \cap C) = P(A \mid C)}$$

equiv. to cond. ind. given C

Thus: if C is known to have occurred, the additional knowledge that B occurs, does not change probability of A.

Be warned that

$$\boxed{\text{independence of } A, B \text{ w.r.t } P(\cdot)} \quad \begin{matrix} \not\Rightarrow \\ \not\Leftarrow \end{matrix} \quad \boxed{\text{conditional independence of } A, B \text{ w.r.t } P(\cdot \mid C)}$$

see examples 1,20, 1,21 in text (p. 37).

Independence of multiple events

What does it mean for 3 events A, B, C to be independent?

Defn

We say A, B, C are independent iff

$$(1)\quad P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$$

$$\begin{cases} (2)\quad P(A \cap B) = P(A) \cdot P(B) \\ (3)\quad P(A \cap C) = P(A) \cdot P(C) \\ (4)\quad P(B \cap C) = P(B) \cdot P(C) \end{cases}$$

(2), (3) and (4) are known as <u>Pairwise independence</u>.

Warning:

$$\boxed{\begin{array}{l}\text{Pairwise}\\ \text{independence}\\ \text{of } A,B,C\\ \text{i.e. (2),(3),(4)}\end{array}} \quad \begin{array}{c}\not\Rightarrow\\ \not\Leftarrow\end{array} \quad \boxed{\begin{array}{l}(1)\ P(A\cap B\cap C)\\ \quad = P(A)\cdot P(B)\cdot P(C)\end{array}}$$

see examples 1.22, 1.23 in text (p. 39).

<u>Exercise</u>

Let $A, B \subseteq \Omega$. Prove that the following are equivalent

(a) A, B are independent

(b) A, B^c " "

(c) A^c, B " "

(d) A^c, B^c " "

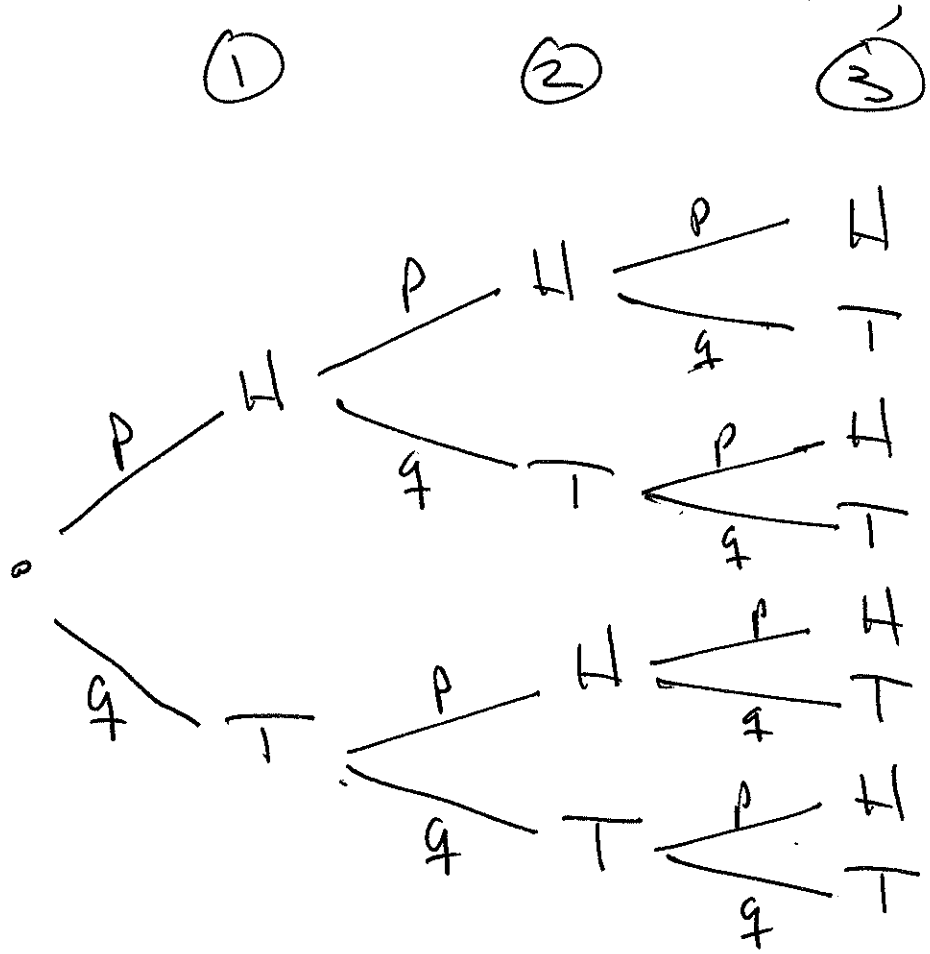
Binomial Probabilities

often an experiment is composed of a seq. of identical, independent **trials**.

A simple experiment with exactly 2 outcomes is called a **Bernoulli trial**.

$$\text{outcomes} = \begin{cases} \text{Success} = 1 = \text{heads} \\ \text{Failure} = 0 = \text{tails} \end{cases}$$

Ex. Perform a seq. 3 independent tosses of a weighted coin, with $P(H) = p$, $P(T) = q = 1-p$



outcome	Prob
HHH	p^3
HHT	p^2q
HTH	p^2q
HTT	pq^2
THH	p^2q
THT	pq^2
TTH	pq^2
TTT	q^3

$$p^3 + 3p^2q + 3pq^2 + q^3 = (p+q)^3 = 1^3 = 1$$

so

$$P(\#\text{heads} = 3) = p^3$$

$$P(\#\text{heads} = 2) = 3p^2 q$$

$$P(\#\text{heads} = 1) = 3p q^2$$

$$P(\#\text{heads} = 0) = q^3$$

Recall! Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

also $\binom{n}{k}$ is called a binomial coefficient

$$\binom{n}{k} = \begin{bmatrix}\text{\# of } k\text{-element subsets} \\ \text{of an } n\text{-element set}\end{bmatrix}$$

$$= \begin{bmatrix}\text{\# of bit strings of} \\ \text{length } n \text{ containing} \\ \text{exactly } k \text{ 1's}\end{bmatrix}$$

$$= \frac{n!}{k!(n-k)!}$$

In general, if we do n independent flips of a weighted coin ($P(H) = p$), then

$$\boxed{P(\#\text{heads} = k) = \binom{n}{k} p^k (1-p)^{n-k}}$$

Binomial Probabilities