CSE 107 9-28-22

supplemental lecture

Recall:

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

can write as

$$P(A\cap B) = P(A|B)\cdot P(B)$$

also

$$P(A\cap B) = P(B|A)\cdot P(A)$$

$$\therefore\ P(A|B)\cdot P(B) = P(B|A)\cdot P(A).$$

Ex.

A test for a certain disease has:

- Positive with Prob. .99, if subject has the disease
- Positive with Prob. .10, if subject does not have disease
- a random individual has disease with Prob. .05

Questions:

- what is the Prob. of false Positive: Positive test and no disease
- what is Prob. of false negative: negative test and has disease.

Let

$T = \{\text{test is Positive}\}$

$D = \{\text{subject has disease}\}$.

We are given

$$P(T \mid D) = .99 \quad \therefore P(T^c \mid D) = .01$$

$$P(T \mid D^c) = .10 \quad \therefore P(T^c \mid D^c) = .90$$

$$P(D) = .05 \quad \therefore P(D^c) = .95$$

we must find

$P(T \cap D^c)$ (false Positive)

$P(T^c \cap D)$ (false negative)

We have

$$P(T \cap D^c) = P(T \mid D^c) \cdot P(D^c)$$
$$= (.10)(.95) = \boxed{.095}$$

$$P(T^c \cap D) = P(T^c \mid D) \cdot P(D)$$
$$= (.01) \cdot (.05) = \boxed{.0005}$$

observe

$$P(A|B)\cdot P(B) = \underbrace{P(A\cap B) = P(B|A)\cdot P(A)}_{\text{Generalize this}}$$

- $P(A\cap B) = P(A)\cdot P(B|A)$
- $P(A\cap B\cap C) = P(A)\cdot P(B|A)\cdot P(C|A\cap B)$ ✓

<u>Proof:</u>

$$RHS = P(\cancel{A})\cdot \frac{P(\cancel{A\cap B})}{P(\cancel{A})}\cdot \frac{P(A\cap B\cap C)}{P(\cancel{A\cap B})}$$

$$= P(A\cap B\cap C) = LHS.$$

- $P(A\cap B\cap C\cap D) = P(A)\cdot P(B|A)\cdot P(C|A\cap B)\cdot P(D|A\cap B\cap C)$

<u>Exercise</u> Prove this

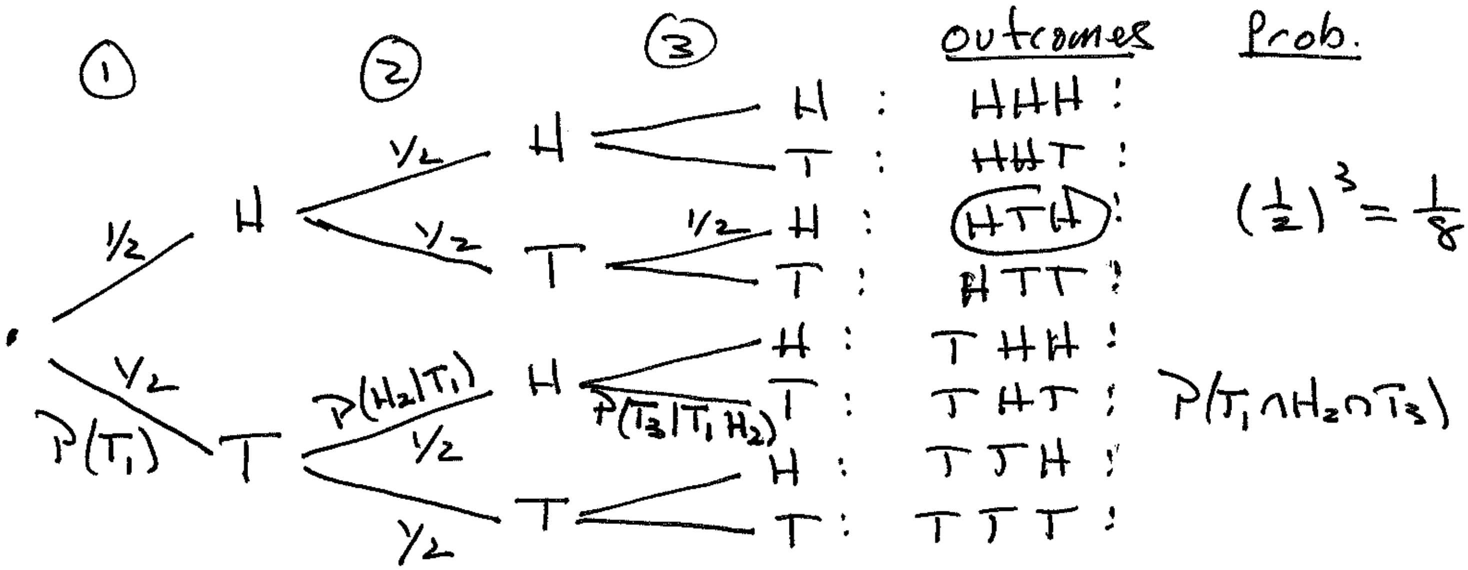
Multiplication rule

$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1)\cdot P(A_2|A_1)\cdot P(A_3|A_1\cap A_2)\cdots$$

$$\cdots P\left(A_n \,\Big|\, \bigcap_{i=1}^{n-1} A_i\right)$$

This rule the common practice of multiplying probabilities down tree diagrams.

Ex. toss 3 fair coins!



conclusion: outcomes are equally likely.

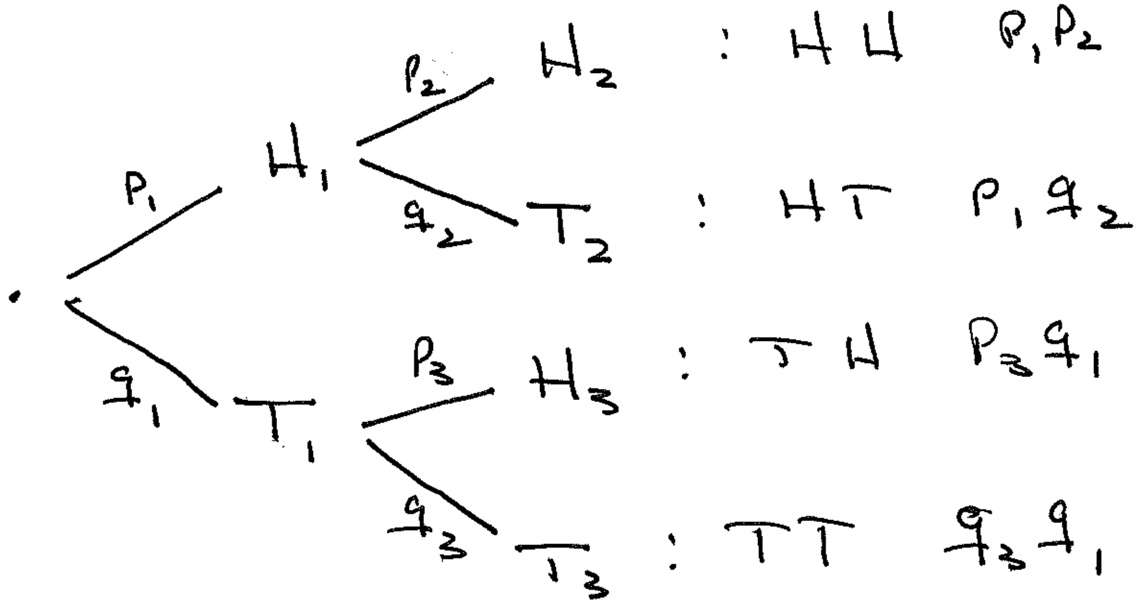
Ex. 3 loaded coins

$P(H_1) = p_1$

$P(T_1) = q_1 = 1 - p_1$

in general $P(H_i) = p_i$, $P(T_i) = q_i = 1 - p_i$

i.e. $p_i + q_i = 1 \quad (i = 1, 2, 3)$

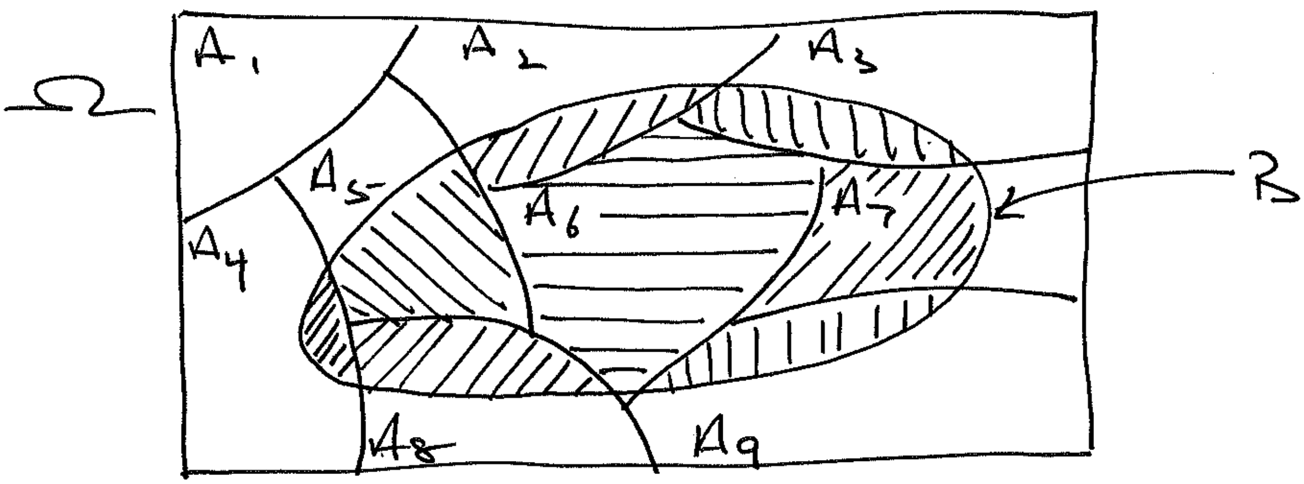


check: $p_1 p_2 + p_1 q_2 + p_3 q_1 + q_3 q_1 = 1$

use $p_1 + q_1 = p_2 + q_2 = p_3 + q_3 = 1$

1.4 Total Probability, Bayes rule

Suppose $A_1, A_2, \ldots, A_n$ form a partition of Ω



note:

$$B = \bigcup_{i=1}^{n} (A_i \cap B)$$

Total Probability Theorem

Assume $P(A_i) > 0$ for $i = 1, 2, \ldots, n$. Then for any $B \subseteq \Omega$, we have

$$\boxed{\begin{aligned} P(B) &= P(A_1 \cap B) + \cdots\cdots + P(A_n \cap B) \\ &= P(A_1) \cdot P(B|A_1) + \cdots\cdots + P(A_n) \cdot P(B|A_n) \end{aligned}}$$

<u>Ex.</u> Roll a fair (6-sided) die. If result is in $\{1,2,3\}$, roll again. otherwise stop.

what is Probability that sum of rolls is ≥ 5.

Let $A_i = \{1^{st}$ roll is $i\}$ $(1 \leq i \leq 6)$.

Then $P(A_i) = \frac{1}{6}$ $(1 \leq i \leq 6)$. Let

$B = \{\text{sum} \geq 5\}$. observe,

- if A_1 occurs, then sum ≥ 5 iff $2^{nd} \in \{4,5,6\}$
- " A_2 " " " " " " $\in \{3,4,5,6\}$
- " A_3 " " " " " " $\in \{2,3,4,5,6\}$

Thus

$$P(B \mid A_1) = \frac{3}{6} = \frac{1}{2}$$

$$P(B \mid A_2) = \frac{4}{6} = \frac{2}{3}$$

$$P(B \mid A_3) = \frac{5}{6}$$

$P(B|A_4) = 0$

$P(B|A_5) = 1$

$P(B|A_6) = 1$

By the total Prob. theorem

$$P(B) = P(B|A_1)\cdot P(A_1) + P(B|A_2)\cdot P(A_2) + \cdots$$
$$\cdots + P(B|A_6)\cdot P(A_6)$$

$$= \frac{1}{2}\cdot\frac{1}{6} + \frac{2}{3}\cdot\frac{1}{6} + \frac{5}{6}\cdot\frac{1}{6} + 0\cdot\frac{1}{6} + 1\cdot\frac{1}{6} + 1\cdot\frac{1}{6}$$

$$= \frac{1}{6}\left(\frac{1}{2} + \frac{2}{3} + \frac{5}{6} + 0 + 1 + 1\right)$$

$$= \frac{1}{6}\left(\frac{24}{6}\right) = \frac{1}{6}\cdot 4 = \boxed{\frac{2}{3}}$$

Baye's Rule

Let $A_1, A_2, \ldots, A_n$ be a Partition of Ω ($A_i \neq \phi$, $A_i \cap A_j = \phi$ if $i \neq j$, $\bigcup_i A_i = \Omega$).

Assume $P(A_i) > 0$ $(1 \leq i \leq n)$ and also $P(B) > 0$. Then

$$P(A_i \mid B) = \frac{P(A_i) \cdot P(B \mid A_i)}{P(B)}$$

$$= \frac{P(A_i) \cdot P(B \mid A_i)}{P(A_1) \cdot P(B \mid A_1) + \cdots\cdots + P(A_n) \cdot P(B \mid A_n)}$$

for any i in $1, 2, \ldots, n$.

To verify this, recall

$$P(B)\cdot P(A|B) = P(A\cap B) = P(A)\cdot P(B|A)$$

$$\therefore \; P(A|B) = \frac{P(A)\cdot P(B|A)}{P(B)}.$$

Apply total probability thm to the denominator.

Baye's rule is used for 'inference' problems. Say we have a number of causes

$$A_1, A_2, \text{----}, A_n$$

that may produce an effect B.

so

$$
\begin{tikzcd}
A_1 \arrow[dr] & \\
A_2 \arrow[r] & B \\
\vdots & \\
A_n \arrow[uur] &
\end{tikzcd}
$$

<u>causes</u> <u>effect</u>

The cond. Prob. $P(B|A_i)$ is a model of the strength of causation

$$A_i \longrightarrow B \;:\; P(B|A_i)$$

<u>causation</u>

Given the effect B is observed we wish to evaluate $P(A_i|B)$

$$A_i \longleftarrow B \;:\; P(A_i|B)$$

<u>inference</u>

we call

$P(A_i|B)$: <u>Posterior Probability</u>

$P(A_i)$: <u>Prior Probability</u>

$P(B|A_i)$: <u>likelihood</u>

$P(B)$: <u>marginal Probability</u>

<u>Bayes rule</u>

Prior likelihood

$$P(A_i|B) = \frac{P(A_i) \cdot P(B|A_i)}{P(B)}$$

Posterior

marginal

Ex recall 'disease' example

$T = \{\text{test positive}\}$

$D = \{\text{has disease}\}$

$P(T|D) = .99$

$P(T|D^c) = .10$

$P(D) = .05 \quad \therefore P(D^c) = .95$

find: P(disease|test +)

i.e. $P(D|T)$. let $A_1 = D$

and $A_2 = D^c$, apply Bayes

theorem.

$$P(D|T) = \frac{P(D) \cdot P(T|D)}{P(T)}$$

$$= \frac{P(D) \cdot P(T|D)}{P(D) \cdot P(T|D) + P(D^c) \cdot P(T|D^c)}$$

$$= \frac{(.05)(.99)}{(.05)(.99) + (.95)(.10)}$$

$$= .34256\ldots \approx \boxed{.3426}$$

round to 4 digits