cse 107 9-27-22

Ex. two fair 6-sided dice are thrown

what is Ω:

SUM= 2 3 4 5 6 7

	11	12	13	14	15	16
	21	22	23	24	25	26
	31	32	33	34	35	36
max =4	41	42	43	44	45	46
	51	52	53	54	55	56
	61	62	63	64	65	66

8 9 10 11 12

min =4

$\{(x,y) \mid x+y=2\}$

1st roll 2nd roll

$P(\text{SUM}=2) = 1/36$

$\ldots =3) = 2/36$

$\ldots =4) = 3/36$

$\ldots =5) = 4/36$

$$P(\text{sum} = 6) = 5/36$$

$$\text{- - - } = 7) = 6/36$$

$$\text{- - - } = 8) = 5/36$$

$$\text{- - - } = 9) = 4/36$$

$$\text{- - - } = 10) = 3/36$$

$$\text{- - - } = 11) = 2/36$$

$$\text{- - - } = 12) = 1/36$$

Also:

$$P(\text{sum is even}) = 2\left(\frac{1}{36} + \frac{3}{36} + \frac{5}{36}\right) = \frac{18}{36} = \frac{1}{2}$$

$$P(\text{sum is odd}) = \frac{1}{2}$$

$$P(\max = 4) = 7/36$$

$$P(\min = 4) = 5/36$$

also:

$$\left.\begin{array}{l} P(1^{st}\text{ roll} > 2^{nd}\text{ roll}) = 15/36 \\ P(1^{st}\text{ roll} = 2^{nd}\text{ roll}) = 6/36 \\ P(\text{at least one } 3) = 11/36 \end{array}\right\} \underline{\underline{\text{verify}}}$$

continuous models

if Ω is some kind of continuum, (like a subset of $\mathbb{R}$, or $\mathbb{R}^2$, or $\mathbb{R}^3$) then the singleton Probabilities $P(\omega)$ where $\omega \in \Omega$, are (in general) not suffient to determine $P(A)$ for all $A \subseteq \Omega$.

Ex.

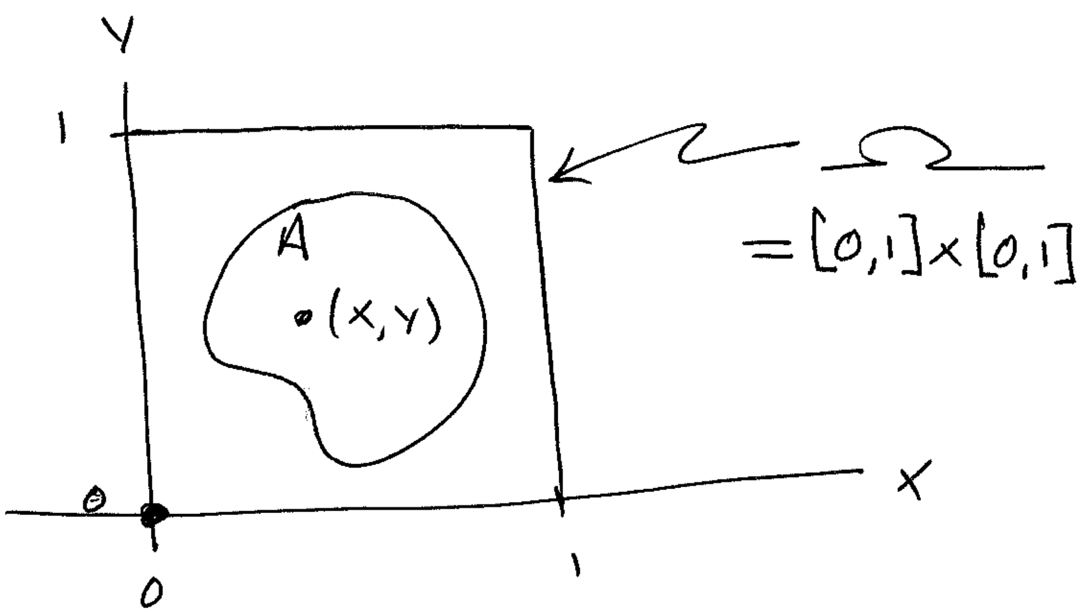
R. & J. have a date. Each will be late by a random time (hours) in $[0, 1]$. The 1st to arrive will wait is min ($\frac{1}{4}$ hour), then leave if the other has not arrived.

find P(they meet)

let x = R's delay time $\in [0,1]$

y = J's " " $\in [0,1]$.

assume all delay pairs (x,y) are equally likely



"Equally likely" means $P(A)$ is proportionally to area(A): $P(A) = c \cdot \text{area}(A)$

so $1 = P(\Omega) = c \cdot \text{area}(\Omega) = c \cdot 1 = c$

$\therefore c = 1$

$\therefore P(A) = \text{area}(A)$.

<u>note</u>: $P((x,y)) = 0$

we seek $P(M)$ where

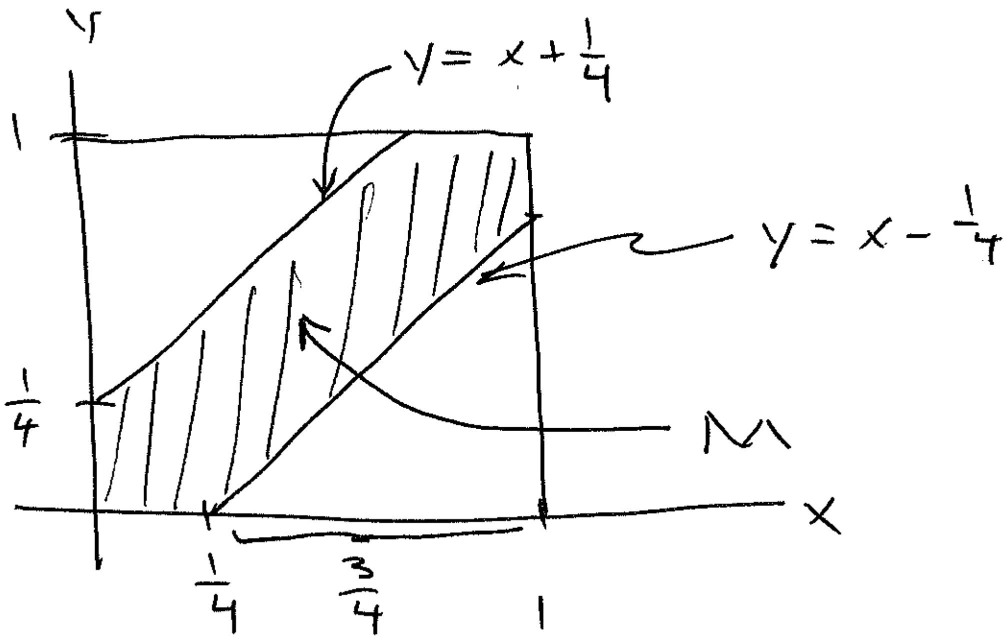
$$M = \{(x,y) \mid |x-y| \le \tfrac{1}{4}, (x,y) \in \Omega\}$$

$$|x-y| \le \tfrac{1}{4}$$

iff $\quad -\frac{1}{4} \le x - y \le \frac{1}{4}$

$$\swarrow \qquad \searrow$$

$y \le x + \frac{1}{4} \qquad y \ge x - \frac{1}{4}$



$$P(M) = 1 - \left(\tfrac{1}{2}\cdot\left(\tfrac{3}{4}\right)^2 + \tfrac{1}{2}\left(\tfrac{3}{4}\right)^2\right)$$

$$= 1 - \frac{9}{16}$$

$$= \frac{7}{16}$$

In general, if $\Omega \subseteq \mathbb{R}^2$, uniform means

$$P(A) = c \cdot \text{area}(A)$$

since $1 = P(\Omega) = c\,\text{area}(\Omega)$

$$\therefore \quad c = \frac{1}{\text{area}(\Omega)}$$

$$\therefore \quad P(A) = \frac{\text{area}(A)}{\text{area}(\Omega)}$$

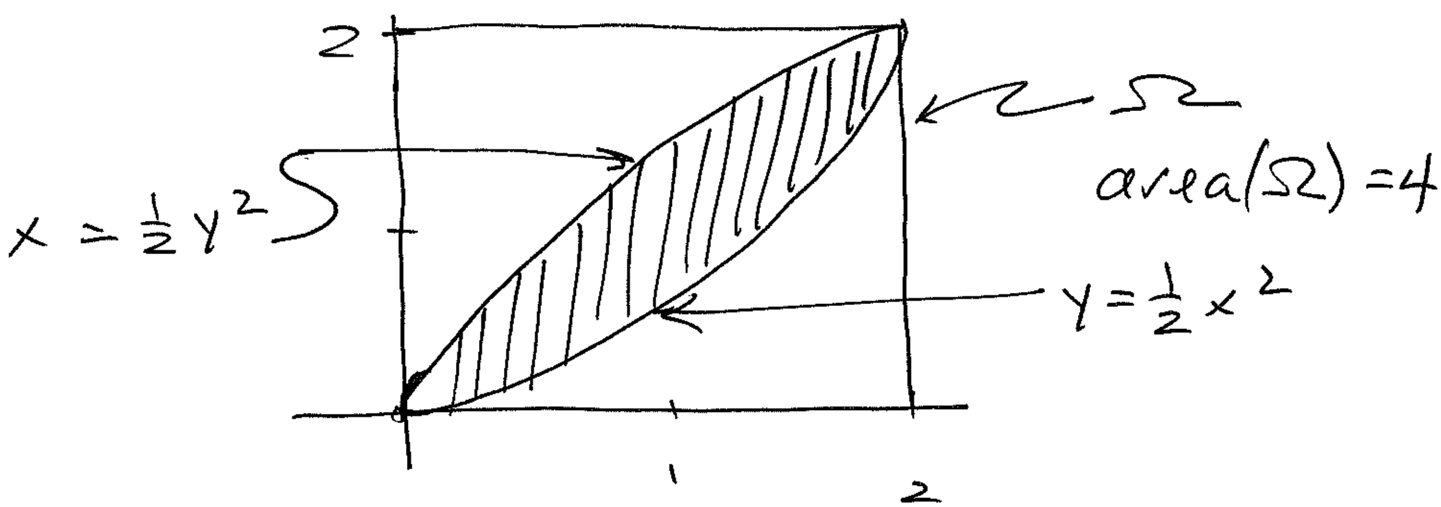
Exercise

Alice Pick's $x \in [0, 2]$ "random"

Bob Pick's $y \in [0, 2]$ "random"

what is Probability that both

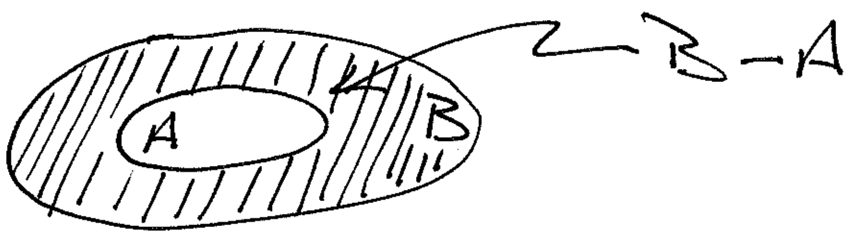
$$\boxed{2x > y^2 \text{ and } 2y > x^2}$$



Properties of $P(\cdot)$

The following facts are consequences of the axioms of Probability.

(a) if $A \subseteq B$, then $P(A) \le P(B)$



Proof:

$$A \subseteq B \implies B = A \cup (B-A)$$

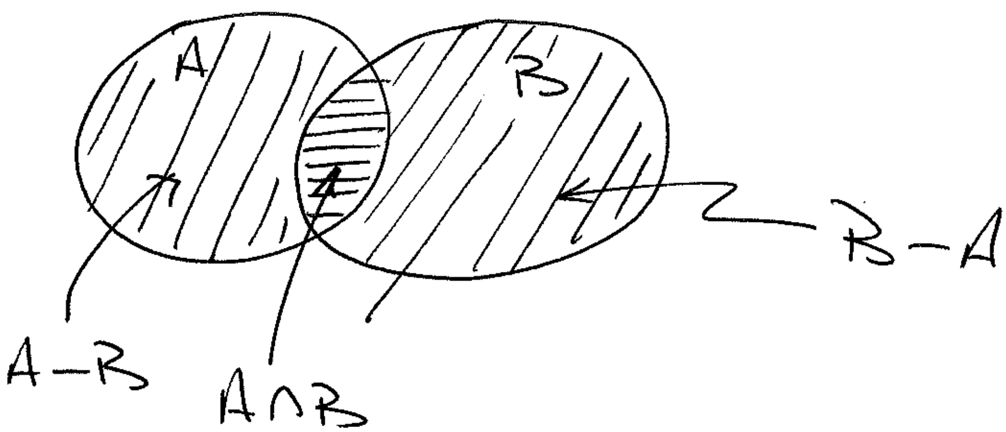
Since $A \cap (B-A) = \phi$, and $P(B-A) \ge 0$, we have:

$$P(B) = P(A) + P(B-A) \ge P(A)$$

$$\therefore \quad P(A) \le P(B).$$

(b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ ✓

Proof:



$$P(A \cup B) = P(A-B) + P(A \cap B) + P(B-A)$$

Also: $P(B) = P(B-A) + P(A \cap B)$

$\therefore P(B-A) = P(B) - P(A \cap B)$

Similarly: $P(A-B) = P(A) - P(A \cap B)$

$$\therefore P(A \cup B) = [P(A) - \cancel{P(A \cap B)}] + \cancel{P(A \cap B)} + [P(B) - P(A \cap B)]$$

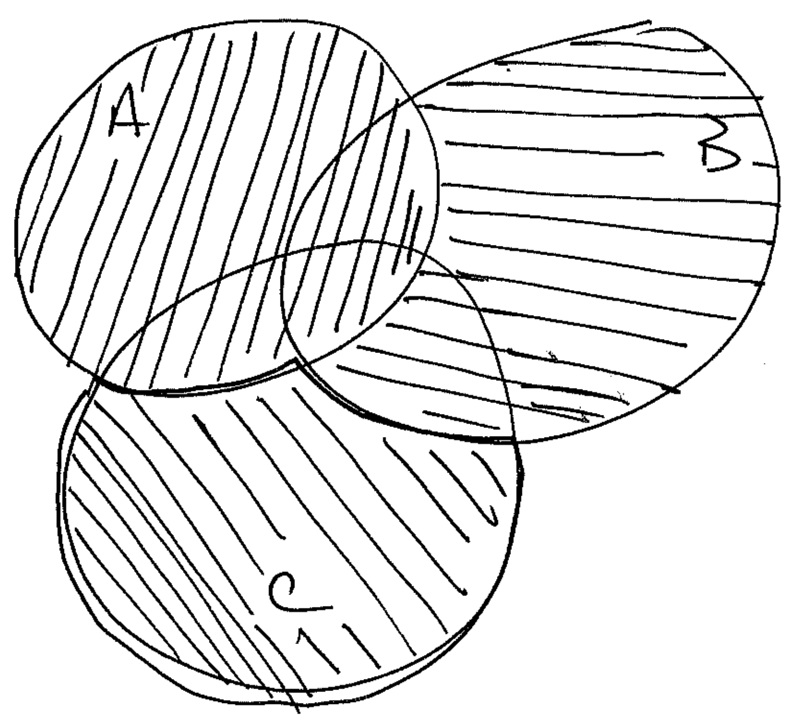
$$= P(A) + P(B) - P(A \cap B)$$

(c) $P(A \cup B) \leq P(A) + P(B)$

follows from (b)

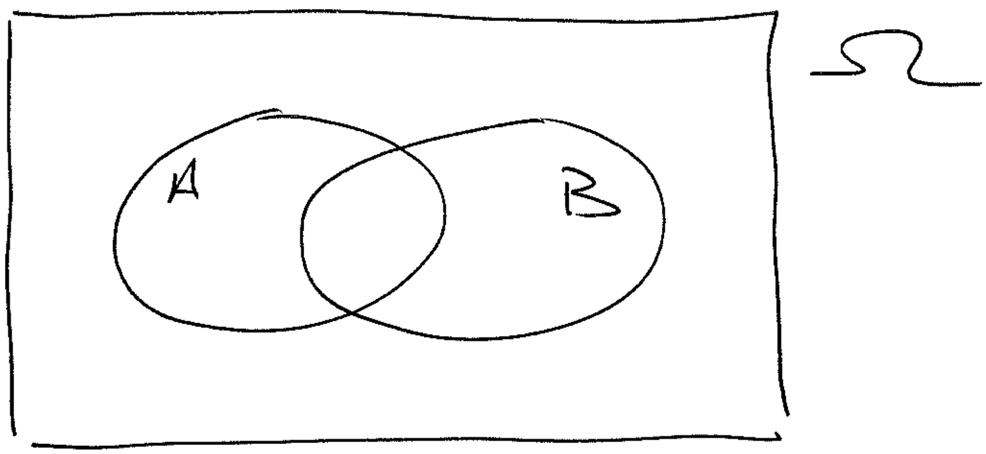
(d) $P(A \cup B \cup C) = P(A) + P(A^c \cap B)$

$+ P(A^c \cap B^c \cap C)$



1.3 Conditional Probability

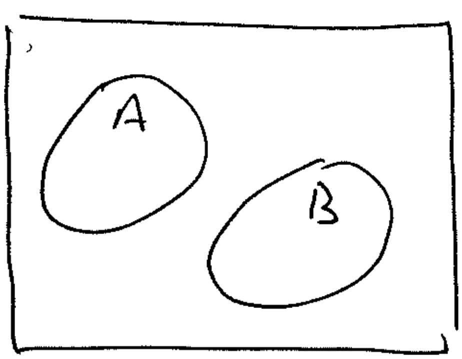
Suppose $A, B \subseteq \Omega$



If we know B occurs, it will in general, affect our belief about whether A occurs.

extreme cases:

①



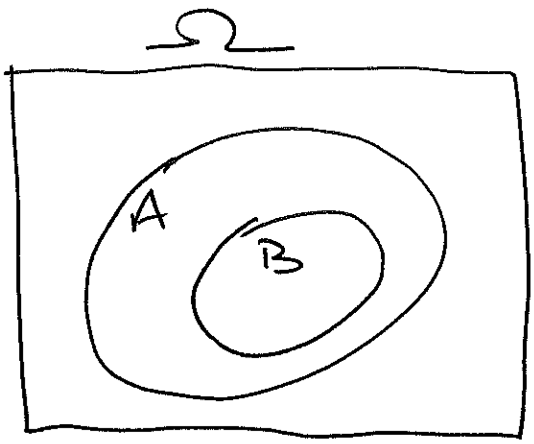
$A \cap B = \emptyset$

$B \subseteq A^c$

$\therefore$ B occurs

$\Rightarrow$ A does not occur.

(2)



$B \subseteq A$

B occurs

$\Rightarrow A$ does occur

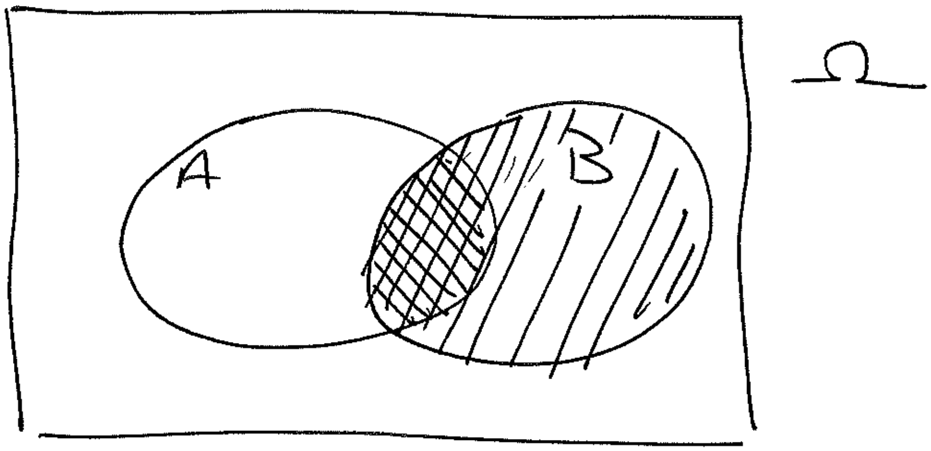
Defn

The conditional Probability of A given B, denoted $P(A|B)$ is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

(where we assume $P(B) > 0$.)

Picture



$P(A|B)$ is the "re-normalized area" of $A \cap B$ in B.

Note: $P(\cdot\,|B)$ is a valid Probability law, in that axioms are satisfied.

(i) $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} \geq 0$

(ii) $P(\Omega \mid B) = \dfrac{P(\Omega \cap B)}{P(B)}$

$= \dfrac{P(B)}{P(B)} = 1$

(iii) if $A_1 \cap A_2 = \phi$, then

$$P(A_1 \cup A_2 \mid B) = \frac{P((A_1 \cup A_2) \cap B)}{P(B)}$$

$$= \frac{P((A_1 \cap B) \cup (A_2 \cap B))}{P(B)}$$

$$= \frac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)}$$

$$= \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)}$$

$$= P(A_1|B) + P(A_2|B)$$

anything that follows from axioms for $P(\cdot)$, also holds for $P(\cdot|B)$

for instance:

$$P(A_1 \cup A_2|B) = P(A_1|B) + P(A_2|B) - P(A_1 \cap A_2|B)$$

<u>also</u>: if Ω is finite, uniform, then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{|A \cap B|/|\Omega|}{|B|/|\Omega|}$$

$$= \frac{|A \cap B|}{|B|}$$

<u>note</u>:

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

can be written

$$P(A\cap B) = P(A|B)\cdot P(B)$$