CSE 107 12-1-22

- Final Exam: Mon. 12-5-22
  4:00 - 6:00 pm
- SETs: close Sun. 12-4-22
  at 11:59 pm.

Sum of independent Poisson r.v.s

Let X, Y be ind. Poisson r.v.s with param. λ_1, λ_2 resp. Then

$$Z = X + Y$$

is Poisson with param. $\lambda_1 + \lambda_2$

Proof.

use convolution to find PMF of Z.

we have

$$P_X(k) = e^{-\lambda_1} \cdot \frac{\lambda_1^k}{k!} \qquad k = 0, 1, \ldots$$

and

$$P_Y(k) = e^{-\lambda_2} \cdot \frac{\lambda_2^k}{k!} \qquad k = 0, 1, \ldots$$

Thus

$$P_Z(n) = \sum_{k=0}^{\infty} P_X(k) \cdot P_Y(n-k)$$

$$= \sum_{k=0}^{n} e^{-\lambda_1} \cdot \frac{\lambda_1^k}{k!} \cdot e^{-\lambda_2} \cdot \frac{\lambda_2^{n-k}}{(n-k)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \cdot \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \cdot \lambda_1^k \lambda_2^{n-k}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \cdot \sum_{k=0}^{n} \binom{n}{k} \lambda_1^k \lambda_2^{n-k}$$

$$= e^{-(\lambda_1+\lambda_2)} \cdot \frac{(\lambda_1+\lambda_2)^n}{n!} \quad \text{for } n = 0, 1 \ldots$$

$\therefore$ Z is Poisson with Param. $\lambda_1+\lambda_2$.

$\blacksquare$

<u>Ex.</u> (6.2 #8)

accidents occur:

- Rush hour: 8-9 am: Poisson Proc. with rate 5/hour.

- During 9-11 am: Poisson Proc. with rate 3/hour.

What is the PMF of the total # of accidents from 8 to 11 am

Soln

let

N_1 = # accidents in 8-9 am

Poisson: $\lambda_1 = 5 \cdot 1 = 5$

N_2 = # accidents in 9-11 am

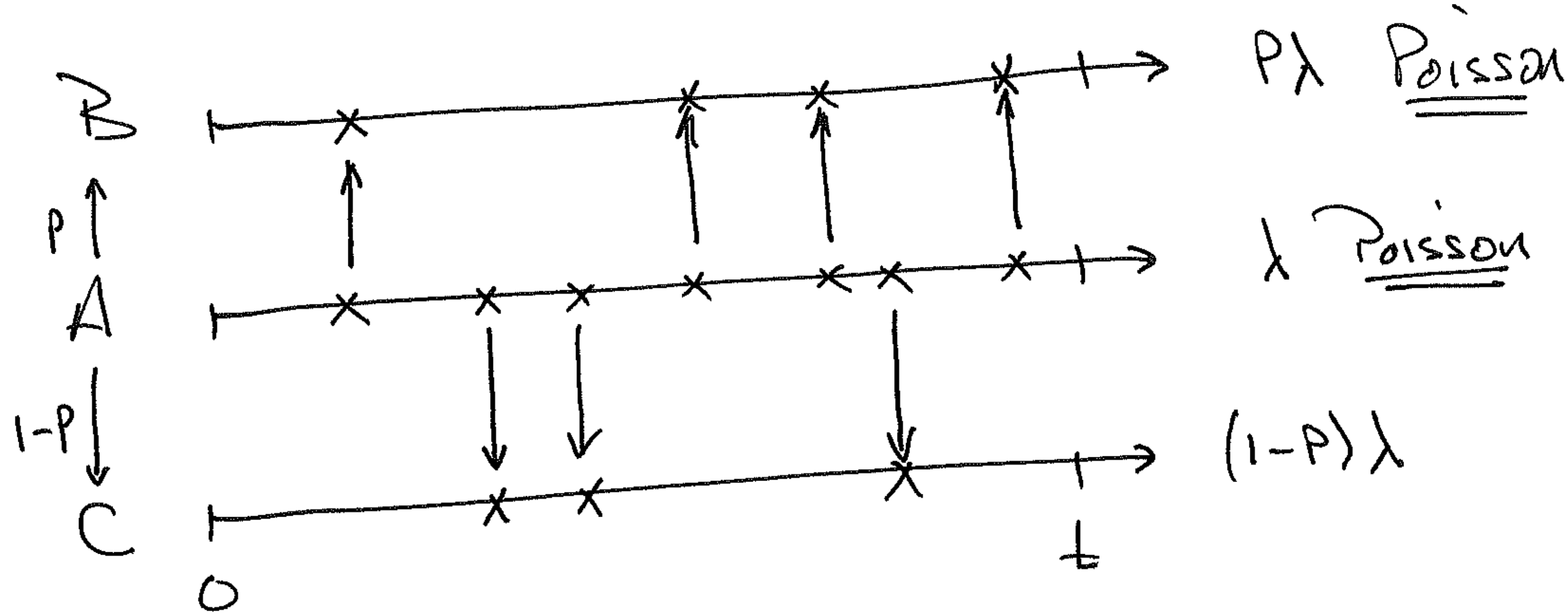
Poisson: $\lambda_2 = 3 \cdot 2 = 6$

$N = N_1 + N_2$ = # accidents in 8-11 am.

Poisson: $\lambda = 5 + 6 = 11$

$$P_N(n) = e^{-11} \cdot \frac{11^n}{n!} \qquad n = 0, 1, \ldots$$

Splitting a Poisson Process



what kind of Processes are B, C

Let

$N = \#$ arrivals of A in $[0,t]$

$X =$ " " " B " "

$Y =$ " " " C " "

We know

$$P_N(n) = e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!} \qquad n = 0, 1, \ldots$$

Then

$$P_X(k) = P(X = k)$$

$$= \sum_{n=k}^{\infty} P(X = k \mid N = n) \cdot P(N = n)$$

$$= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \cdot e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!}$$

$$= \sum_{n=k}^{\infty} \frac{\cancel{n!}}{k!(n-k)!} \cdot p^k (1-p)^{n-k} \cdot e^{-\lambda t} \cdot \frac{(\lambda t)^n}{\cancel{n!}}$$

$$= \frac{p^k}{k!} \cdot e^{-\lambda t} \cdot \sum_{n=k}^{\infty} \frac{(1-p)^{n-k} (\lambda t)^n}{(n-k)!}$$

Sub. $n \leftarrow n+k$

$$= \frac{p^k}{k!} \cdot e^{-\lambda t} \cdot \sum_{n=0}^{\infty} \frac{(1-p)^n \cdot (\lambda t)^{n+k}}{n!}$$

$$= \frac{(p\lambda t)^k}{k!} e^{-\lambda t} \cdot \sum_{n=0}^{\infty} \frac{(1-p)^n \cdot (\lambda t)^n}{n!}$$

$$= \frac{(p\lambda t)^k}{k!} \cdot e^{-\lambda t} \cdot \sum_{n=0}^{\infty} \frac{((1-p)(\lambda t))^n}{n!}$$

$$= \frac{(p\lambda t)^k}{k!} \cdot e^{-\cancel{\lambda t}} \cdot e^{\cancel{\lambda t} - p\lambda t}$$

$$= e^{-p\lambda t} \cdot \frac{(p\lambda t)^k}{k!} \quad \text{for } k = 0, 1, 2 \ldots$$

so X is Poisson with param.

$p\lambda t$.

so B satisfies axioms of a Poisson Process of rate $p\lambda$

Similarly, C is a Poisson Process of rate $(1-p)\lambda$.

<u>Ex.</u> See 6.2 #11

Merging Poisson Processes

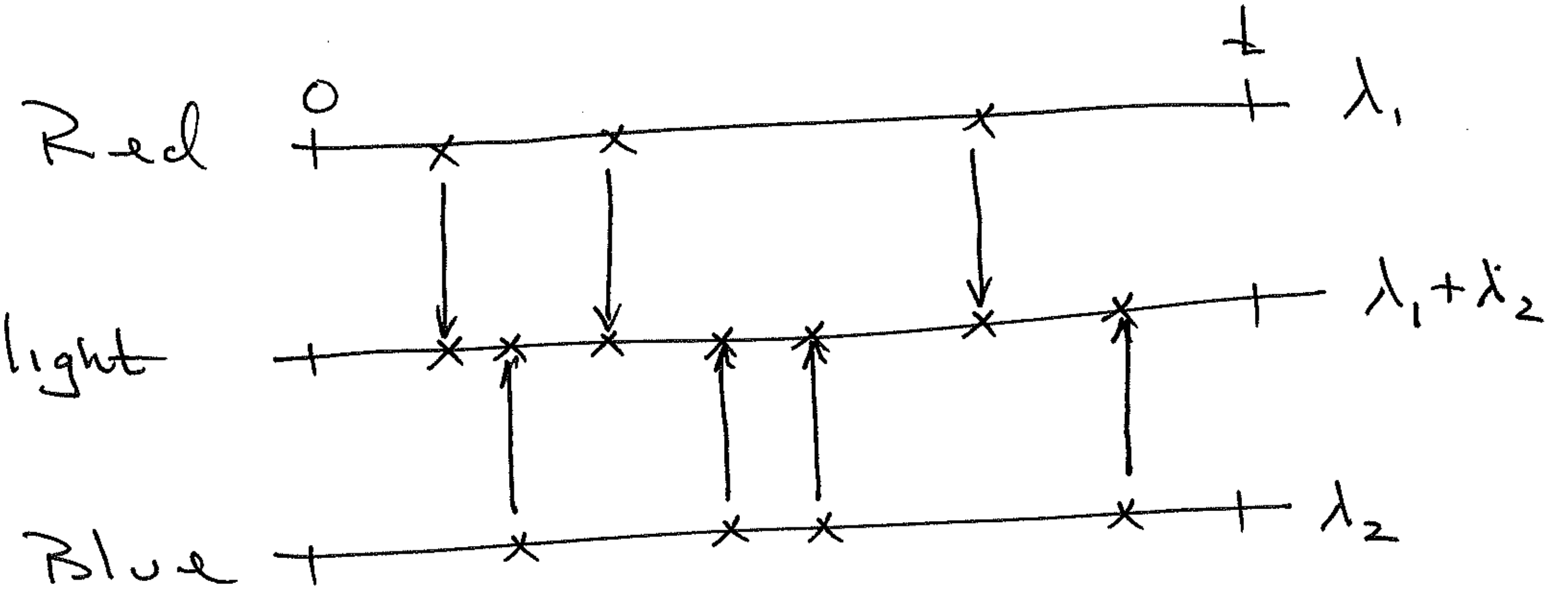
An observer sees two lights flashing.

Red: Poisson rate λ_1

Blue: Poisson rate λ_2

The observer is colorblind, so just sees flashing lights.

What kind of process does observer see?



let X = # Red lights in $[0,t]$

Y = # Blue " " "

M = # lights " "

we know: X is Poisson $(\lambda_1 t)$

Y " " $(\lambda_2 t)$

$\therefore$ $N = X + Y$ is also Poisson

with Parameter $\lambda_1 t + \lambda_2 t = (\lambda_1 + \lambda_2)t$

$\therefore$ light seen by observer is a Poisson Process with rate $\lambda_1 + \lambda_2$.

Ex.

Suppose observer sees a single flash, what is the probability that it was real?

Soln.

Let $\delta > 0$ be len. of a small time interval containing the flash. We seek

$$P(1 \text{ Red} \mid 1 \text{ light})$$

$$= \frac{P(1 \text{ Red}, 1 \text{ light})}{P(1 \text{ light})}$$

$$= \frac{P(1 \text{ Red})}{P(1 \text{ light})} = \frac{\lambda_1 \delta}{(\lambda_1 + \lambda_2)\delta}$$

$$= \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

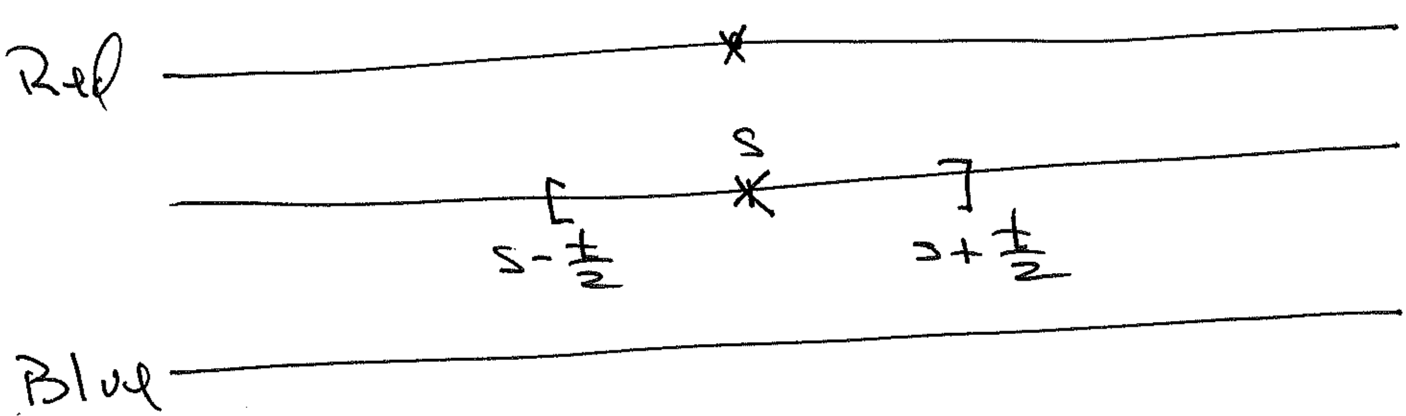
similarly of Blue given a light flash

$$P(1 \text{ Blue} \mid 1 \text{ light}) = \frac{\lambda_2}{\lambda_1 + \lambda_2}.$$

Alternate solution:

Suppose light flashes at time s.
Consider an interval

$$I_t = \left[s - \frac{t}{2}, s + \frac{t}{2}\right]$$



Then

$$P(1 \text{ Red in } I_t \mid 1 \text{ light } I_t)$$

$$= \frac{P(1 \text{ Red in } I_t)}{P(1 \text{ light in } I_t)}$$

$$= \frac{e^{-\lambda_1 t} \cdot \frac{(\lambda_1 t)^1}{1!}}{e^{-(\lambda_1+\lambda_2)t} \cdot \frac{(\lambda_1+\lambda_2)t}{1!}}$$

$$= e^{\lambda_2 t} \cdot \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right) \xrightarrow[t \to 0]{\text{as}} \frac{\lambda_1}{\lambda_1+\lambda_2}$$

5.1 Markov & Chebyshev ineq.

Markov inequality

Let X be a r.v. taking only non-negative values. Then

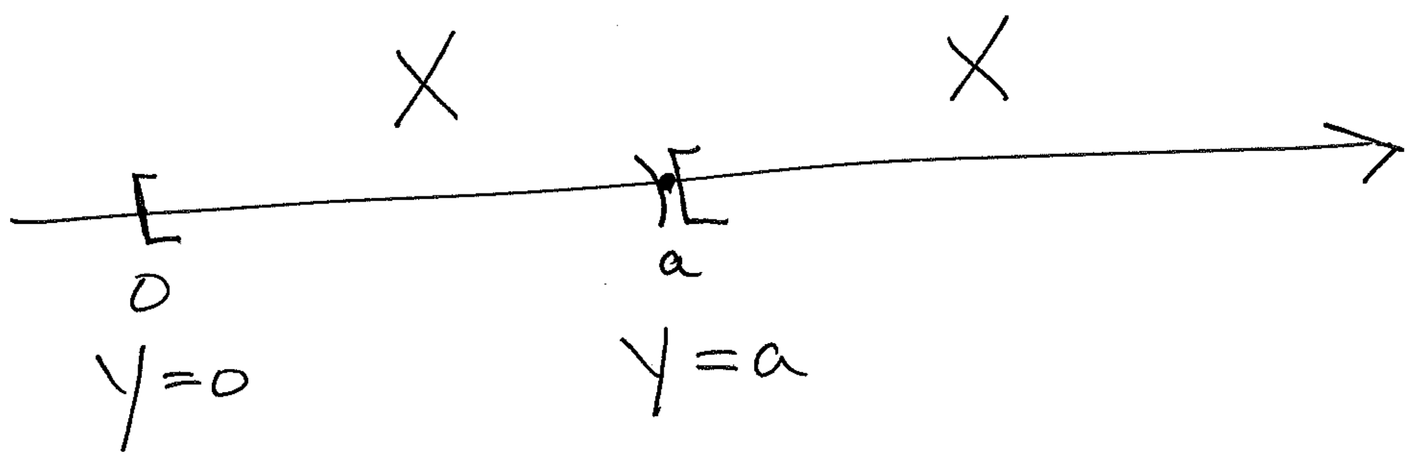
$$P(X \geq a) \leq \frac{E[X]}{a}$$

for any $a > 0$.

Proof.

fix $a > 0$. define.

$$Y = \begin{cases} 0 & \text{if } 0 \le X < a \\ a & \text{if } X \ge a \end{cases}$$



Thus $Y \le X$.

$\therefore \quad E[Y] \le E[X]$

Also

$$E[Y] = 0 \cdot P(Y=0) + a \cdot P(Y=a)$$

$$= a \cdot P(X \ge a).$$

$$\therefore \quad a\,P(X \geq a) \leq E[X]$$

$$\therefore \quad P(X \geq a) \leq \frac{E[X]}{a}. \qquad \blacksquare$$