- mid2 : Tue NOV. 15

Observe that

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y)\,dx = \int_{-\infty}^{\infty} \frac{f_{X,Y}(x,y)}{f_Y(y)}\,dx$$

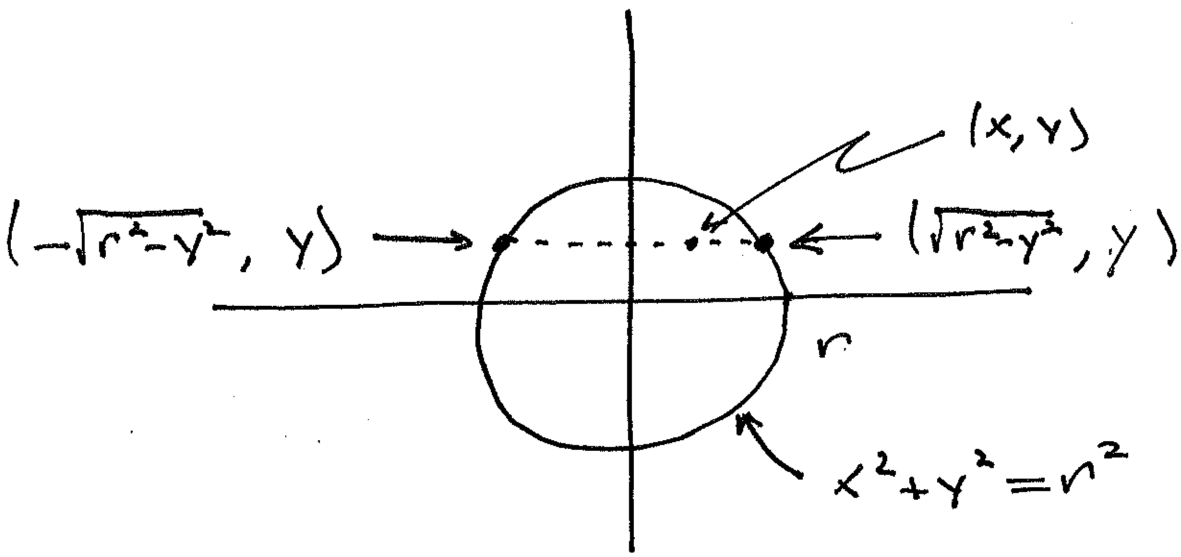
$$= \frac{\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx}{f_Y(y)}$$

$$= \frac{f_Y(y)}{f_Y(y)} = 1$$

Thus $f_{X|Y}(x|y)$ is a valid PDF.

Ex.

throw a dart at a circular target of radius r.



Assume target is always hit, otherwise & point (x,y) is uniform. Let X, Y be coordinates of hit point. find

$$f_{X,Y}(x,y),\ f_Y(y) \text{ and } f_{X|Y}(x|y)$$

We are given

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{\pi r^2} & x^2+y^2 \le r^2 \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{1}{\pi r^2}\,dx$$

$$= \begin{cases} \dfrac{2\sqrt{r^2-y^2}}{\pi r^2} & \text{if } |y| \le r \\ 0 & \text{otherwise} \end{cases}$$

finally

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \begin{cases} \dfrac{1}{2\sqrt{r^2-y^2}} & x^2+y^2 \le r^2 \\ 0 & \text{otherwise} \end{cases}$$

All same identities that hold for PMFs also hold for PDFs!

- $f_{X|Y}(x|y) f_Y(y) = f_{X,Y}(x,y)$
  $= f_{Y|X}(y|x) \cdot f_X(x)$

- multiplication rule

$$f_{X,Y,Z}(x,y,z) = f_{X|Y,Z}(x|y,z) \cdot f_{Y|Z}(y|z) \cdot f_Z(z)$$

Conditional Expectation

Defns

- $$E[X|A] = \int_{-\infty}^{\infty} x f_{X|A}(x)\, dx$$

- $$E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx$$

expected value rule

- $E[g(X)|A] = \int_{-\infty}^{\infty} g(x) f_{X|A}(x)\, dx$

- $E[g(X)|Y=y] = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y)\, dx$

total expectation theorem

- if $A_1, \ldots, A_n$ form a partition of Ω, and $P(A_i) > 0$ (all i), then

(1) $$E[X] = \sum_{i=1}^{n} P(A_i) E[X|A_i]$$

- also if $f_Y(y) > 0$ for all y

$$(2)\quad E[X] = \int_{-\infty}^{\infty} f_Y(y) \cdot E[X \mid Y=y]\, dy$$

<u>Proof of (1)</u>:

start with total prob. thm

$$f_X(x) = \sum_{i=1}^{n} P(A_i) \cdot f_{X|A_i}(x)$$

mult by x, integrate w.r.t x, $-\infty$ to ∞

$$\int_{-\infty}^{\infty} x f_X(x)\, dx = \int_{-\infty}^{\infty} x \sum_{i=1}^{n} P(A_i) f_{X|A_i}(x)\, dx$$

$$= \sum_{i=1}^{n} P(A_i) \int_{-\infty}^{\infty} x f_{X|A_i}\, dx$$

$$\therefore \quad E[X] = \sum_{i=1}^{n} P(A_i)\, E[X \mid A_i]$$

∎

<u>Proof of (2)</u>:

start with RHS:

$$\int_{-\infty}^{\infty} E[X \mid Y=y]\, f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx \right) f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y) \cdot f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \cdot f_{X,Y}(x,y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} x \left(\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \right) dx$$

$$= \int_{-\infty}^{\infty} x\, f_X(x)\, dx = E[X].$$ ∎

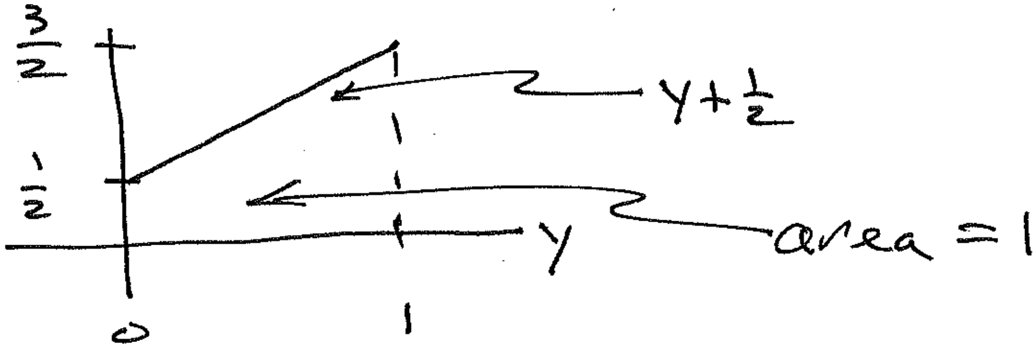
<u>note</u>: Partition here is

$$A_y = \{Y = y\} \text{ for } y \in \text{range}(Y)$$

<u>EX</u>

let X, Y be jointly continuous with

$$f_Y(y) = \begin{cases} y + \frac{1}{2} & 0 \le y \le 1 \\ 0 & \text{other} \end{cases}$$



and

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{x+y}{y+\frac{1}{2}} & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{other} \end{cases}$$

Thus

$$E[X \mid Y = y] = \int_{-\infty}^{\infty} x \left(\frac{x+y}{y+\frac{1}{2}} \right) dx$$

$$= \int_0^1 x \left(\frac{x+y}{y+\frac{1}{2}} \right) dx$$

$$= \cdots\cdots = \frac{\frac{1}{2}y + \frac{1}{3}}{y + \frac{1}{2}}$$

for $0 \le y \le 1$

so

$$E[X] = \int_{-\infty}^{\infty} E[X|Y=y] \cdot f_Y(y)\,dy$$

$$= \int_0^1 \left(\frac{\frac{1}{2}y+\frac{1}{3}}{\cancel{y+\frac{1}{2}}}\right)\cancel{(y+\tfrac{1}{2})}\,dy$$

$$= \cdots\cdots = \boxed{\frac{7}{12}}$$

another way:

$$f_{X,Y}(x,y) = f_{X|Y}(x|y) \cdot f_Y(y)$$

$$= \begin{cases} x+y & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{other} \end{cases}$$

so

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= \int_0^1 (x+y)\,dy = \cdots = \begin{cases} x+\frac{1}{2} & 0 \le x \le 1 \\ 0 & \text{other} \end{cases}$$

and

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\,dx$$

$$= \int_0^1 x\left(x+\tfrac{1}{2}\right)dx = \cdots = \boxed{\frac{7}{12}}$$

■

Note:

$$E[X \mid Y=y] = \begin{cases} \dfrac{\frac{1}{2}y+\frac{1}{3}}{y+\frac{1}{2}} & 0 \le y \le 1 \\ 0 & \text{otherwis} \end{cases}$$

call this fcn $g(y)$.

Define a new r.v.

$$g(Y) \underset{\text{defn}}{=} E[X|Y]$$

Exercise:

$$E[g(Y)] = E[E[X|Y]] = \boxed{\frac{7}{12}}$$

we'll see in chap. 4 that, in general

$$E[E[X|Y]] = E[X]$$

Independence

let X, Y be jointly cont. r.v.s

Defn

we say X, Y are <u>independent</u> iff

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$$

for all $x \in range(X)$, $y \in range(Y)$.

Equivalently:

$$f_{X|Y}(x|y) = f_X(x)$$ if $f_Y(y) > 0$ and all x, y.

also

$$f_{Y|X}(y|x) = f_Y(y)$$ for $f_X(x) > 0$ all y & x.

If X, Y are ind., then any definable in terms of X, Y are also ind.

$$\{X \in A\} \text{ and } \{Y \in B\}$$

for $A, B \subseteq \mathbb{R}$. In Particular

$$F_{X,Y}(x,y) = P(X \le x, Y \le y)$$

$$= \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(s,t)\, ds\, dt$$

$$= \int_{-\infty}^{y} \int_{-\infty}^{x} f_X(s) \cdot f_Y(t)\, ds\, dt$$

$$= \int_{-\infty}^{x} f_X(s)\, ds \cdot \int_{-\infty}^{y} f_Y(t)\, dt$$

$$= P(X \leq x) \cdot P(Y \leq y)$$

$$= F_X(x) \cdot F_Y(y).$$

<u>note</u>: converse is true, i.e.

if $F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y)$,

then X, Y are independent.

As in the discrete case, we have for independent cont. r.v.s

(1) $E[XY] = E[X] \cdot E[Y]$

(2) $E[g(X) \cdot h(Y)] = E[g(X)] \cdot E[h(Y)]$

(3) $Var(X+Y) = Var(X) + Var(Y)$

<u>Exercise:</u>

Prove these

(1) follows from definition

(2) adapt Problem #44 (ch. 2, P. 134)

(3) mimic notes 10-20-22 P. 6-8.

3.6 Baye's Rule

Recall original Baye's rule

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

if $A_1, \ldots, A_n$ form a Partition of Ω, we use total Probability

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i)$$

$$= \sum_{i=1}^{n} P(B|A_i) \cdot P(A_i)$$

we get (for any k with $1 \le k \le n$)

$$\boxed{P(A_k|B) = \frac{P(B|A_k) \cdot P(A_k)}{\sum_{i=1}^{n} P(B|A_i) \cdot P(A_i)}}$$

if X, Y are discrete r.v.s, then

$$\boxed{p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x) \cdot p_X(x)}{\sum_t p_{Y|X}(y|t) \cdot p_X(t)}}$$

In the continuous case this is.

$$\boxed{f_{X|Y}(x|y) = \frac{f_X(x)\cdot f_{Y|X}(y|x)}{\int_{-\infty}^{\infty} f_X(t)\, f_{Y|X}(y|t)\, dt}}$$