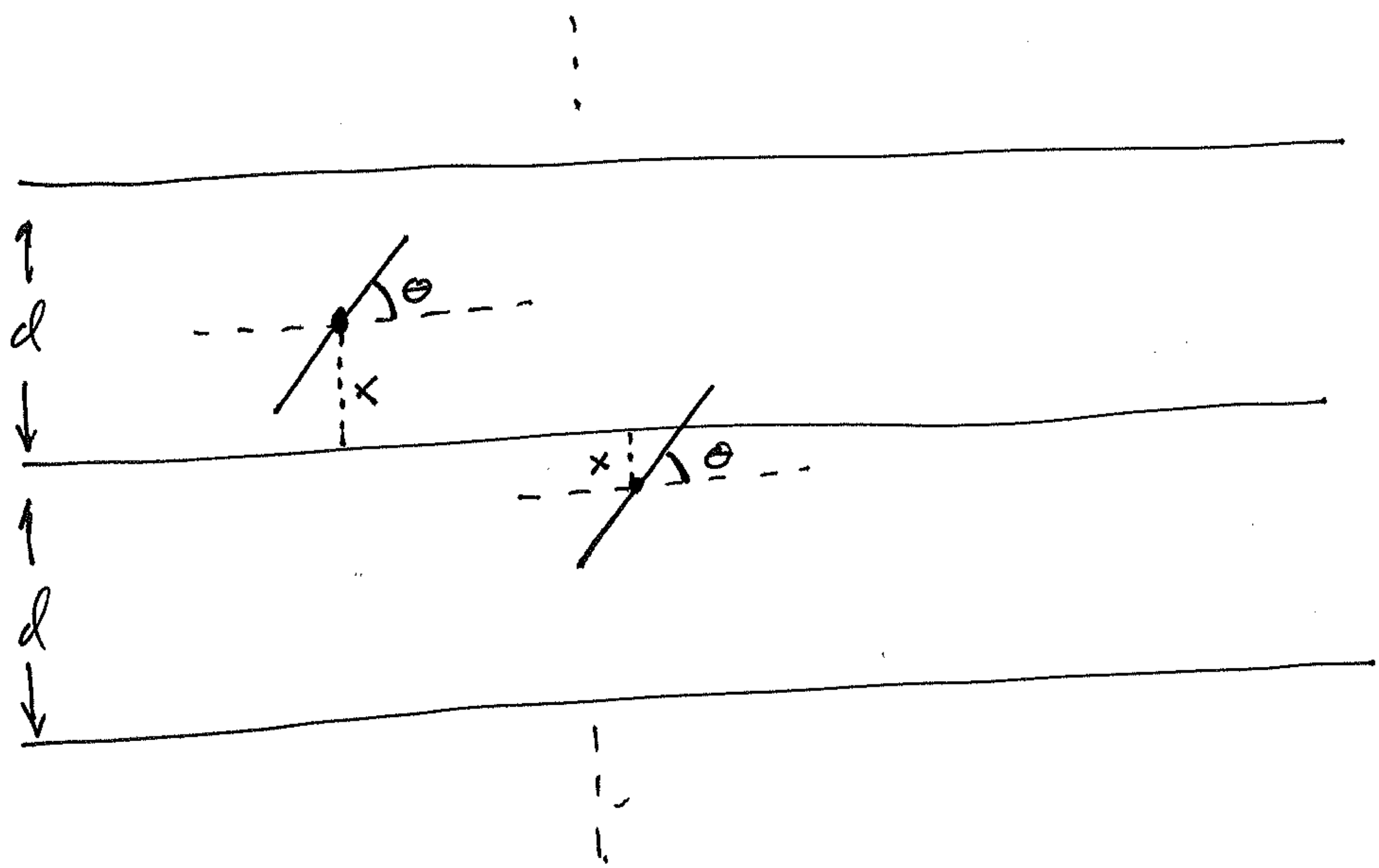
cse 107 11-3-22

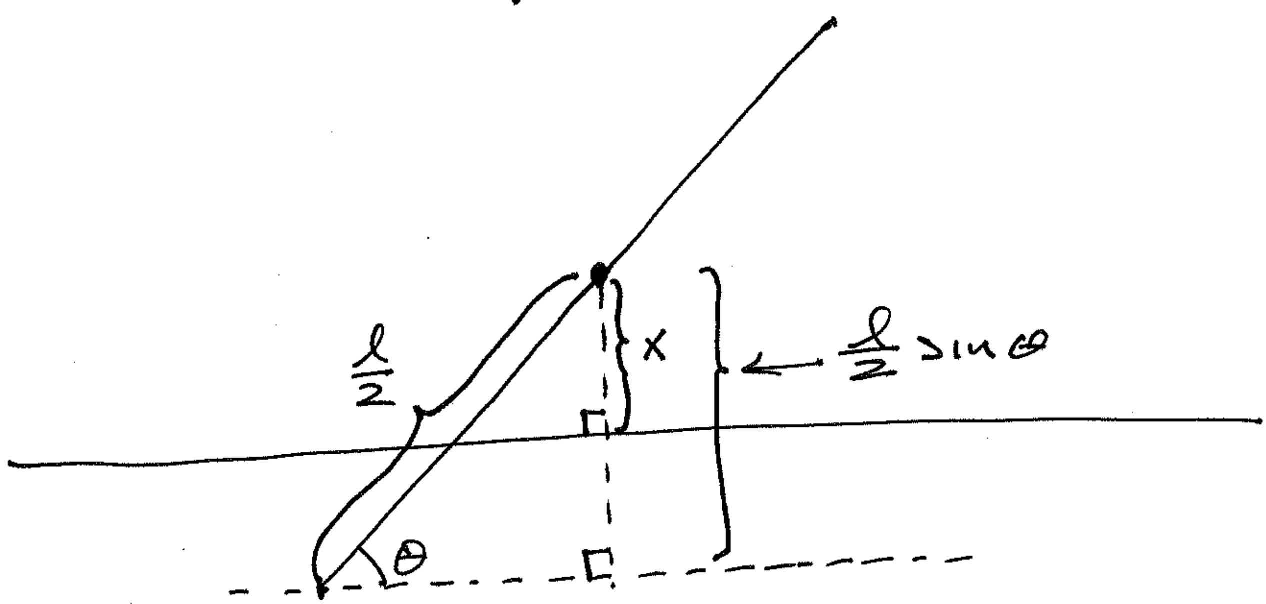
Buffon's needle .....

Let X be the vertical distance of the needle's midpoint to nearest line. let $\Theta$ be the acute angle made by the needle & horizontal.



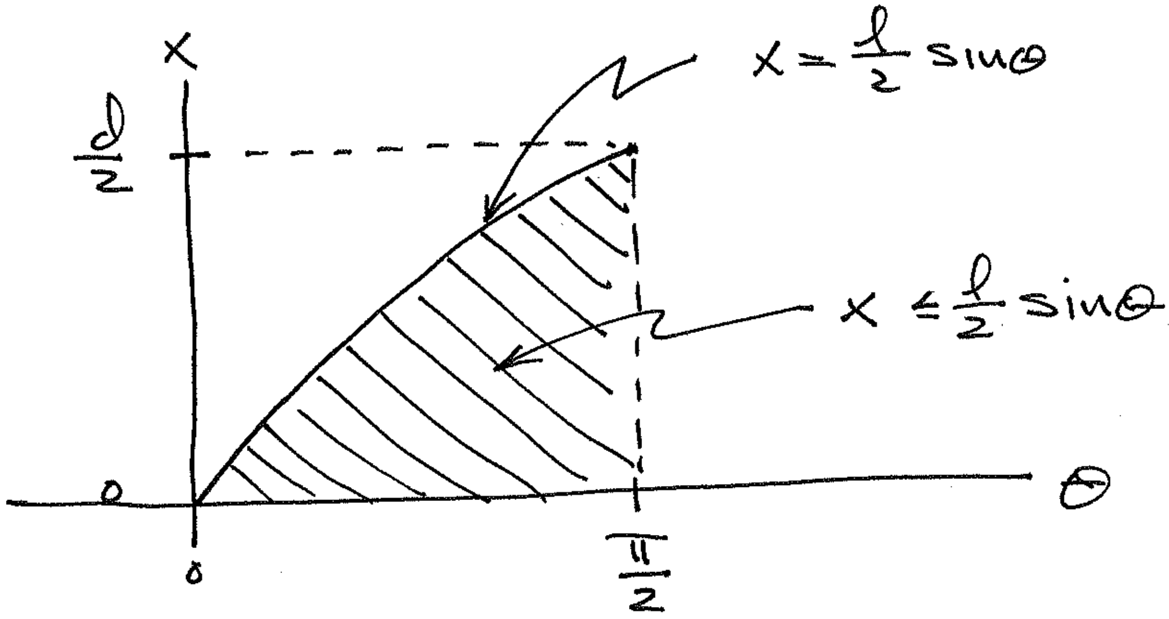
model $(X, \Theta)$ as joint uniform PDF on $[0, \frac{d}{2}] \times [0, \frac{\pi}{2}]$, area $= \frac{\pi d}{4}$

$$f_{X,\Theta}(x, \theta) = \begin{cases} \frac{4}{\pi d} & 0 \le x \le \frac{d}{2},\ 0 \le \theta \le \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$




needle intersects iff

$$X \le \frac{\ell}{2} \sin \Theta$$




$$P\left(X \le \frac{l}{2}\sin\theta\right) = \iint\limits_{x \le \frac{l}{2}\sin\theta} f_{X,\Theta}(x,\theta)\,dx\,d\theta$$

$$= \frac{4}{\pi d}\int_0^{\frac{\pi}{2}}\int_0^{\frac{l}{2}\sin\theta} 1 \cdot dx\,d\theta$$

$$= \frac{4}{\pi d}\cdot\frac{l}{2}\int_0^{\pi/2}\sin\theta\,d\theta$$

$$= \frac{2l}{\pi d}\left(-\cos\theta\right)\Big|_0^{\pi/2} = \frac{2l}{\pi d}\Big(-0-(-1)\Big) = \frac{2l}{\pi d}$$

If we pick $l = d$, then

$$P(\text{intersect}) = \frac{2}{\pi}$$

hence

$$\pi = \frac{2}{P(\text{intersect})}$$

Identify Prob. with rel. frequency, then

$$\pi \approx \frac{2}{\frac{\#\text{intersect}}{\#\text{tosses}}} = \frac{2 \cdot \#\text{tosses}}{\#\text{intersections}}$$

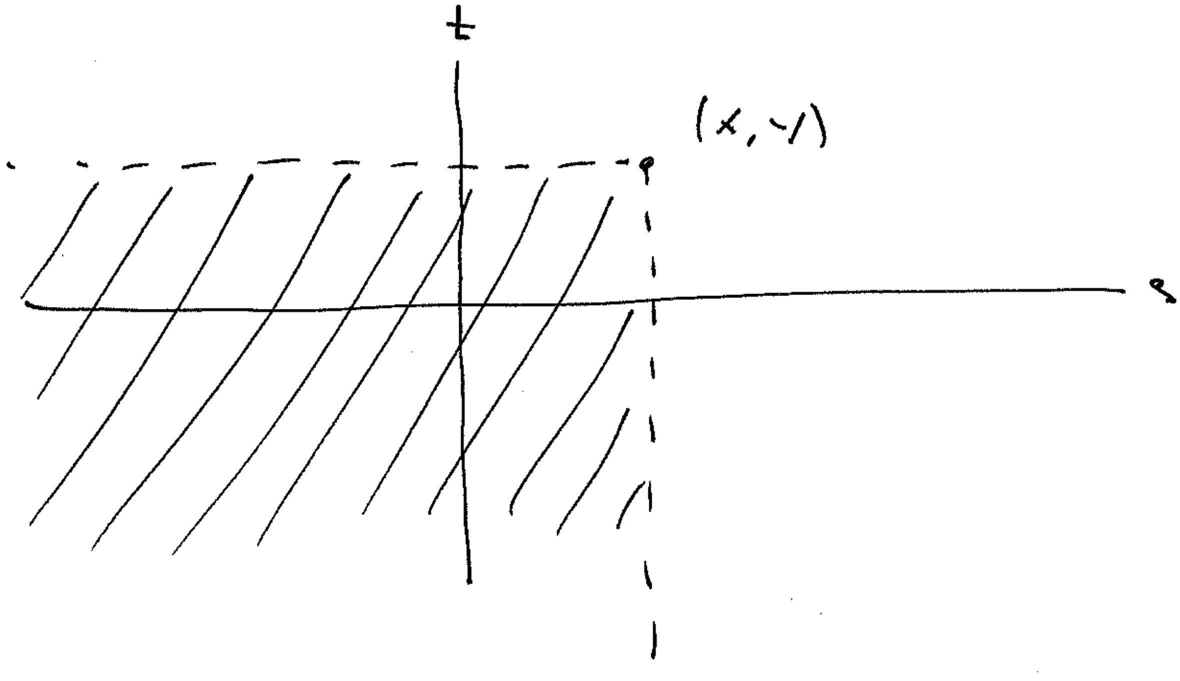
## Joint CDFs

if $X, Y$ two jointly continuous r.v.s

their joint CDF

$$F_{X,Y}(x,y) = P(X \le x, Y \le y)$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s,t)\, dt\, ds$$



can recover joint PDF by

$$f_{X,Y}(x,y) = \frac{\partial^2 F}{\partial y \partial x}(x,y)$$

<u>Expectation</u>

If $g : \mathbb{R}^2 \to \mathbb{R}$, we have the 2-variable expected value rule

$$E[g(X,Y)] = \iint_{\mathbb{R}^2} g(x,y) f_{X,Y}(x,y)\, dx\, dy$$

special case

$$E[aX+bY+c] = aE[X]+bE[Y]+c$$

Proof.

$$E[aX+bY+c]$$

$$= \iint_{\mathbb{R}^2} (ax+by+c)\, f_{X,Y}(x,y)\, dx\, dy$$

$$= a\int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy\, dx$$

$$+ b\int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy$$

$$+ c\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy$$

$$= a\int_{-\infty}^{\infty} x\, f_X(x)\, dx + b\int_{-\infty}^{\infty} y\, f_Y(y)\, dy + c$$

$$= a\,E[X] + b\,E[Y] + c.$$

## more than 2 r.v.s

Given $X, Y, Z$

Joint PDF: $f_{X,Y,Z}(x,y,z)$

Joint CDF: $F_{X,Y,Z}(x,y,z)$

have relations!

$$f_{X,Y}(x,y) = \int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z)\,dz$$

$$f_X(x) = \int_{-\infty}^{\infty}\int_{\infty}^{\infty} f_{X,Y,Z}(x,y,z)\,dy\,dz$$

and expected value rules:

$$E[aX+bY+cZ]=aE[X]+bE[Y]+cE[Z]$$

Conditioning

Begin by conditioning on a single event $A$, with $P(A)>0$. Let $X$ be a cont. r.v.

Defn

The conditional PDF of $X$ given $A$ is a function $f_{X|A}(x)$ such that

$$P(X\in B|A)=\int_B f_{X|A}(x)\,dx$$

for 'any' subset $B\subseteq\mathbb{R}$.

we require:

$$\int_{-\infty}^{\infty} f_{X|A}(x)\,dx = 1$$

special case: event $A$ is of the form

$$A = \{X \in C\}$$

where $C \subseteq \mathbb{R}$. Assume $P(X \in C) > 0$

Then for any $B \subseteq \mathbb{R}$:

$$P(X \in B \mid X \in C) = \frac{P(X \in B, X \in C)}{P(X \in C)}$$

$$= \frac{P(X \in B \cap C)}{P(X \in C)}.$$

$$= \frac{\int_{B \cap C} f_X(x)\, dx}{P(X \in C)}$$

$$= \int_{B \cap C} \frac{f_X(x)}{P(X \in C)}\, dx$$

compare this to defn of cond. PDF

$$f_{X|X\in C}(x) = \begin{cases} \dfrac{f_X(x)}{P(X \in C)} & \text{if } x \in C \\ 0 & \text{otherwise} \end{cases}$$

Ex.

The time $T$ until a new light bulb burns out is Exponential with parameter $\lambda$. Alice turns on the light, leaves and returns $t$ hours later, finding light still on. So the event

$$A = \{T > t\}$$

has occurred. Let $X$ be the additional time until burn out.

Find conditional PDF

$$f_{X|A}(x)$$

Soln.

first find conditional CDF, defined as

$$F_{X|A}(x) = P(X \le x \mid A)$$

$$= \int_{-\infty}^{x} f_{X|A}(s)\, ds ,$$

then differentiate to get

$$\frac{d}{dx} F_{X|A}(x) = f_{X|A}(x) .$$

Recall from 3.1 that

$$P(T > a) = \begin{cases} e^{-\lambda a} & \text{if } a > 0 \\ 1 & a \leq 0 \end{cases}$$

(notes 10-27-22, p. 3)

Thus for $x \geq 0$:

$$P(X > x \mid A) = P(T > t + x \mid T > t)$$

$$= \frac{P(T > t+x, T > t)}{P(T > t)}$$

$$= \frac{P(T > t+x)}{P(T > t)}$$

$$= \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}} = \frac{\cancel{e^{-\lambda t}} \cdot e^{-\lambda x}}{\cancel{e^{-\lambda t}}} = e^{-\lambda x}$$

$$\therefore\ P(X > x \mid A) = \begin{cases} e^{-\lambda x} & x \geq 0 \\ 1 & x < 0 \end{cases}$$

$$\therefore\ F_{X|A}(x) = P(X \leq x \mid A)$$

$$= 1 - P(X > x \mid A)$$

$$= \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Hence

$$f_{X|A}(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Thus the exponential r.v. is memoryless.

If events $A_1, A_2, \ldots, A_n$ form a Partition of $\Omega$, the total Prob. thm gives

$$P(X \le x) = \sum_{i=1}^{n} P(A_i) \cdot P(X \le x | A_i)$$

Thus

$$F_X(x) = \sum_{i=1}^{n} P(A_i) \, F_{X|A_i}(x)$$

so diff. both sides:

$$f_X(x) = \sum_{i=1}^{n} P(A_i) f_{X|A_i}(x)$$

as in Discrete case.

Also as in Discrete case, can condition on another r.v. $Y$.

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

we think of $y$ as fixed and $x$ as variable.