CSE 107 11-29-22

- SETs : if response rate $\geq 85\%$ will inc. overall score by 0.1%

6.2 Poisson Process

Recall : Bernoulli Process

seq. $X_1, X_2, \ldots$

of Bernoulli r.v.s satisfying

(1) each X_t has param. p

(2) the set $\{X_1, X_2, \ldots\}$ is independent

defined

$$Y_k = \text{time to } k^{th} \text{ arrival}$$

found Y_k was Pascal (p, k)

Also defined

$$T_1 = Y_1$$

$$T_k = Y_k - Y_{k-1} \qquad k = 2, 3, \ldots$$

Each T_k is geometric (p).

we can also define

$$N_t = X_1 + X_2 + \cdots + X_t \qquad t = 1, 2, 3, \ldots$$

i.e. the # of arrivals up to time t. note N_t is Binomial.

$$P_{N_t}(k) = \binom{t}{k} p^k (1-p)^{t-k}$$

we can recover X_t from N_t as:

$$X_1 = N_1$$

$$X_t = N_t - N_{t-1} \qquad t = 2, 3, \ldots$$

can also recover

$$Y_k = \min\{t \mid N_t = k\}$$

so any one of: X_t, N_t, Y_k, T_k could be used to define a Bernoulli Process.

The Poisson Process is the cont. time analog of Bernoulli Process.

Consider a cont. time arrival Process, and define

$$P(k,\tau) = P\left(\text{there are } k \text{ arrivals in an interval of len. } \tau\right)$$

<u>Defn</u>

An arrival Process is called a <u>Poisson Process</u> with rate $\lambda > 0$ iff it has properties.

(a) <u>time-homogeneity</u>

$P(k,\tau)$ is the same for all time intervals of length τ.

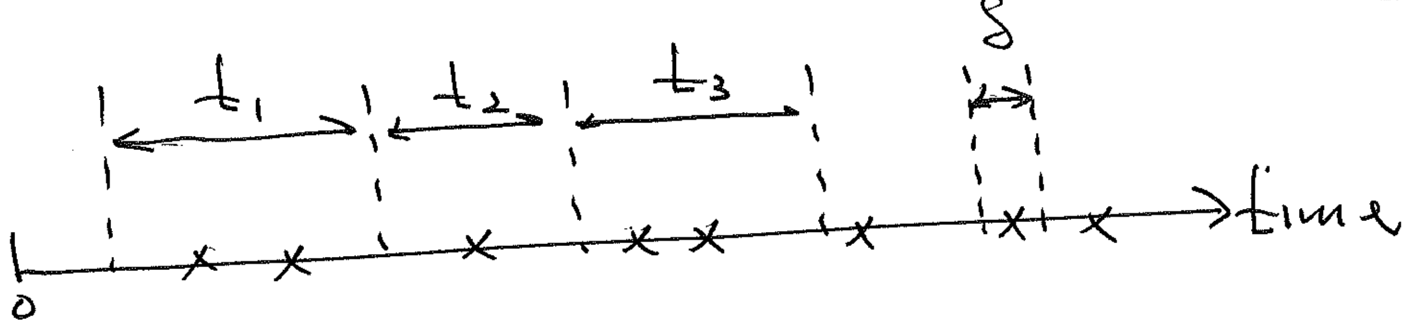
(b) <u>independence</u>

the # of arrivals in disjoint time intervals are independent

(c) <u>small interval Probabilities</u>

let $\delta > 0$ be small. Then

$$P(k,\delta) \approx \begin{cases} 1-\lambda\delta & \text{if } k=0 \\ \lambda\delta & \text{if } k=1 \\ 0 & \text{if } k=2,3,\ldots \end{cases}$$



The approximation in (c) refers to limit as $\delta \to 0$. more precisely

$$P(k,\delta) = \left\{ \begin{array}{ll} 1-\lambda\delta & k=0 \\ \lambda\delta & k=1 \\ 0 & k\geq 2 \end{array} \right\} + o(\delta)$$

where $o(\delta)$ is a fcn. with property

$$\lim_{\delta \to 0} \frac{o(\delta)}{\delta} = 0 .$$

Ex. $c \cdot \delta^2$ has this property.

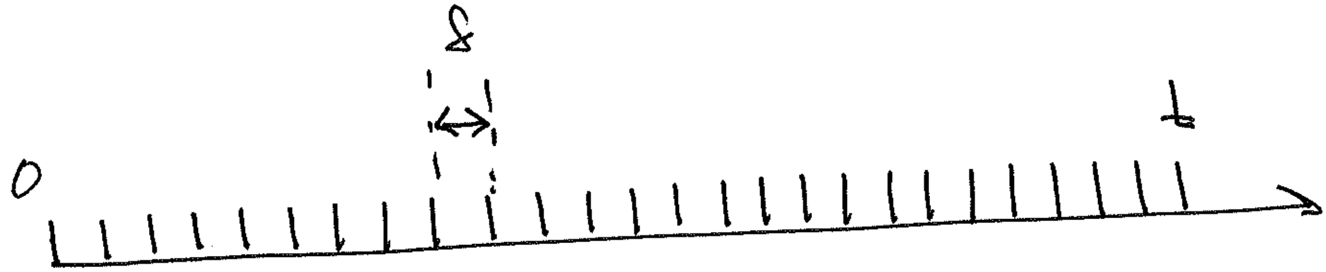
note:

$$E[\#\text{arrivals in } [t, t+\delta]]$$

$$= (1-\lambda\delta)\cdot 0 + \lambda\delta\cdot 1 = \lambda\delta$$

$\therefore$ λ is # arrivals/unit time

What about # arrivals in large intervals.



let $n = \frac{t}{\delta}$, the # of subintervals

choose δ so small that Prob. of more than 1 arrival is negligible.

We now have, approximately, a Bernoulli Process with Prob. of arrival

$$p = \lambda \delta = \frac{\lambda t}{n}$$

approx. improves as $\delta \to 0$

The # arrivals in n trials in a Bernoulli is Binomial (n, p). So approx.

P (k arrivals in $[0, t]$)

$$= \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \binom{n}{k} \left(\frac{\lambda t}{n}\right)^k \left(1 - \frac{\lambda t}{n}\right)^{n-k}$$

as $\delta \to 0$, we have $n \to \infty$

and by (handout on B.-P. limit)

we have

P(k arrivals in $[0,t]$)

$$= e^{-\lambda t} \cdot \frac{(\lambda t)^k}{k!}$$

Let N_t denote the # of arrivals in $[0,t]$ of the Poisson Process.

Then

$$\boxed{P_{N_t}(k) = e^{-\lambda t} \cdot \frac{(\lambda t)^k}{k!}}$$

also

$$E[N_t] = \lambda t$$

and

$$Var(N_t) = \lambda t$$

Let T be time to 1^{st} arrival.

we derive dist. of T. observe

$$\{T > t\} = \{N_t = 0\}.$$

Thus

$$F_T(t) = P(T \le t)$$

$$= 1 - P(T > t)$$

$$= 1 - P(N_t = 0)$$

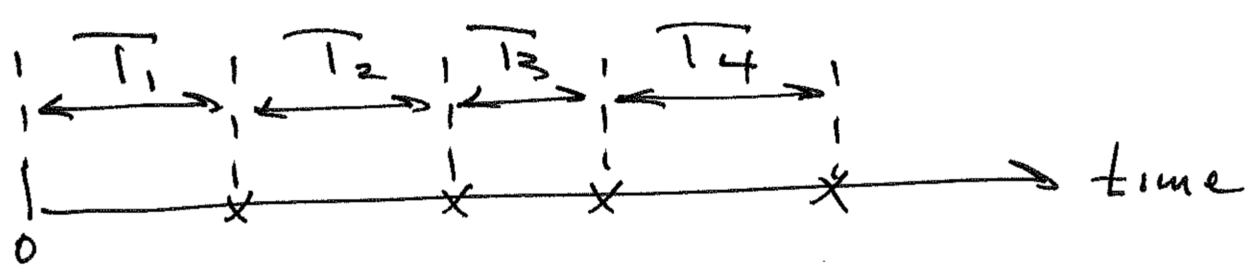
$$= 1 - P_{N_t}(0)$$

$$= 1 - e^{-\lambda t}$$

$$\therefore \quad f_T(t) = \lambda e^{-\lambda t} \quad \text{for } t \geq 0$$

Thus T is exponential with parameter λ.

Let T_k be the k^{th} inter-arrival time



after each arrival, the Poisson process has a 'fresh start', so each T_K is exponential with parameter λ. also $T_1, T_2 \ldots$ are independent.

$$f_{T_K}(t) = \lambda e^{-\lambda t} \quad \text{for } t \geq 0.$$

now define

$$Y_K = T_1 + T_2 + \cdots + T_K \qquad K = 1, 2, \ldots$$

i.e. Y_K is time to K^{th} arrival

what is dist. of Y_K ?

<u>Erlang PDF of order K:</u>

$$f_{Y_K}(t) = \frac{\lambda^K t^{K-1} e^{-\lambda t}}{(K-1)!} \quad \text{for } t \in [0, \infty)$$

<u>Exercise</u>

Prove that the sum of K ind. exponential(λ) r.v.s is Erlang(λ, K)

<u>Hint</u> use induction on K, when $K=1$, Erlang <u>is</u> exponential.

use convolution on induction step to show

$$Y_K = Y_{K-1} + T_K$$

is Erlang of order K.

Summarize Bernoulli vs. Poisson Process:

	Bernoulli	Poisson
time:	$t = \{1, 2, \ldots\}$	$t \in [0, \infty)$
rate:	p / trial	λ / unit time
N_t:	Binomial(t, p)	Poisson(λt)
T_k:	Geometric(p)	Exponential(λ)
Y_k:	Pascal(p, k)	Erlang(λ, k)

Ex.

your email arrives according to a Poisson Proc. at rate

$$\lambda = 2 \text{ mess.}/\text{hour}$$

you currently have no messages.

a.) what's prob. of finding 3 new messages if you check in 30 minutes.

$$P(N_{\frac{1}{2}} = 3) = P_{N_{\frac{1}{2}}}(3)$$

$$= e^{-(2 \cdot \frac{1}{2})} \cdot \frac{(2 \cdot \frac{1}{2})^3}{3!}$$

$$= e^{-1} \cdot \frac{1}{6} = \boxed{.0613}$$

b.) what's Prob of finding no message in 30 min?

$$P(N_{\frac{1}{2}}=0) = e^{-(2\cdot\frac{1}{2})} \cdot \frac{(2\cdot\frac{1}{2})^0}{0!}$$

$$= e^{-1} = \boxed{.3679}$$

(c) What expected time to 5^{th} new message.

$$E[Y_5] = E[T_1 + \cdots + T_5]$$

$$= 5 \cdot E[T]$$

$$= 5 \cdot \frac{1}{\lambda} = 5/2 = \boxed{2.5 \text{ hours}}$$