cse 107 11-22-22

- supplemental lecture: Friday 11-25-22
- " " : last Saturday 11-19-22
- hw7: due Fri. 11-25-22.
- lab 4: due Tue. 11-29-22

6.1 The Bernoulli Process

Defn

The Bernoulli Process is a infinite seq. of ind. Bernoulli r.v.s

$$X_1, X_2, X_3, \ldots$$

Discrete time:

$$\underline{10111110101011101111101} \longrightarrow \text{time}$$

$P(X_t=1) = p$ — language: heads, success, arrival

$P(X_t=0) = 1-p$ — tails, failure, no arrival

Two views

- one experiment, repeated ∞-many times: $\Omega = \{0, 1\}$.

* • a single experiment with

$$\Omega = \{\infty \text{ sequences of } 0\text{'s} \;\&\; 1\text{'s}\}$$

say $\omega = 1111 \ldots\ldots$ (all 1's).

we have

$$P(\omega) = P(\forall t : X_t = 1)$$

$$\leq P(X_t = 1 \text{ for } 1 \leq t \leq n)$$

$$= p^n \quad (\text{for any } n \geq 1)$$

$$p^n \longrightarrow 0 \text{ as } n \to \infty$$

$\therefore$

$$P(\omega) = 0.$$

likewise $P(\omega)=0$ for any $\omega \in \Omega$.

Two basic questions:

① Given n trials (ticks of discrete time) how many successes?

answer: Binomial(n,p)

② how many trials until 1st success

answer: Geometric(p).

Memorylessness

let T be time until 1^{st}

Success. $\therefore T$ is geom. (p)

$$P_T(t) = (1-p)^{t-1} \cdot p$$

$$E[T] = \frac{1}{p}$$

$$Var(T) = \frac{1-p}{p^2}$$

Suppose we observe

$$X_1, X_2, \ldots, X_n$$

all failures. then event

$\{T > n\}$ has occured.

what is dist. until the remaining time to 1^{st} success.

$$P(T-n=t \mid T>n)$$

$$= \frac{P(T-n=t, T>n)}{P(T>n)}$$

$$= \frac{P(T=n+t, T>n)}{P(T>n)}$$

$$= \frac{P(T=n+t)}{P(T>n)} \quad \text{since } \{T=n+t\} \subseteq \{T>n\}$$

$$= \frac{(1-p)^{n+t-1} \cdot p}{(1-p)^{n}} = (1-p)^{t-1} \cdot p$$

$$= P(T=t).$$

Interarrival times

let

T_1 = time to 1^{st} arrival

T_2 = time from T_1 to 2^{nd} arrival

T_3 = " " T_2 " 3^{rd} "

⋮

T_k = " " T_{k-1} " k^{th} ".

let

$$Y_k = T_1 + T_2 + \cdots + T_k$$

= time to k^{th} arrival.

Alternately:

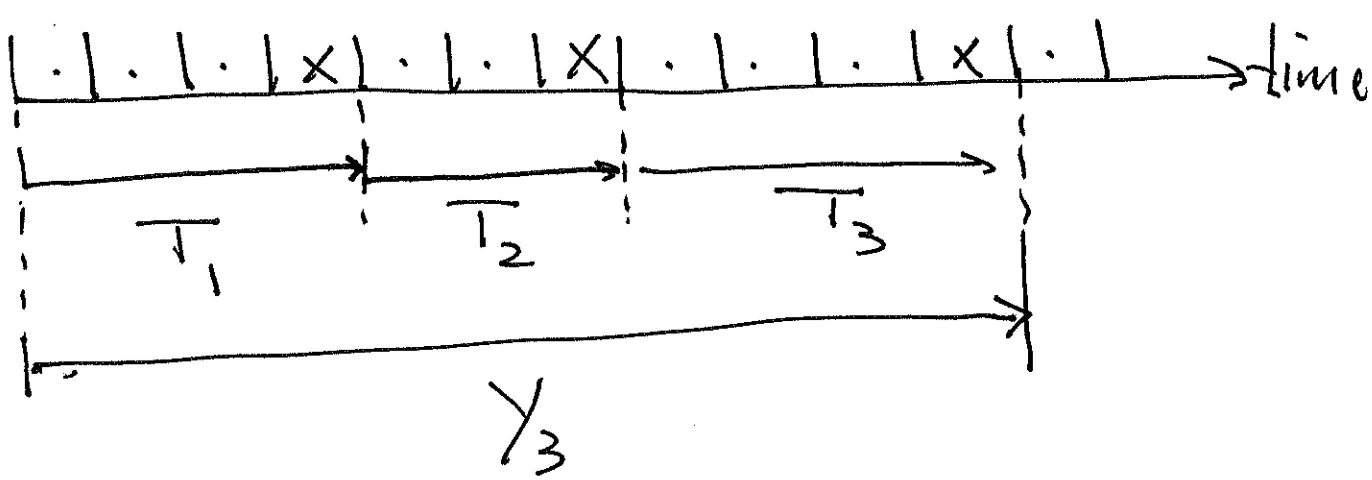
Y_k = time to k^{th} arrival

$T_1 = Y_1$

$T_k = Y_k - Y_{k-1}$ for $k = 2, 3, \ldots$

$\uparrow$

<u>interarrival times</u>



Could get PMF $p_{Y_k}(t)$ from

$Y_k = T_k + Y_{k-1}$ and convolution.

<u>Exercise</u> do this.

<u>shortcut</u>: use independence

$$p_{Y_k}(t) = P(Y_k = t)$$

$$= P(k-1 \text{ arrivals in } [1, t-1], \text{ and } 1 \text{ arrival at time } t)$$

$$= \underbrace{P(k-1 \text{ arr. in } [1,t-1])}_{\text{Binomial}(t-1,\,p,\,k-1)} \cdot \underbrace{P(\text{arr. at } t)}_{p}$$

$$= \binom{t-1}{k-1} \cdot p^{k-1} \cdot (1-p)^{t-k} \cdot p$$

$$= \begin{cases} \binom{t-1}{k-1} p^{k} (1-p)^{t-k} & \text{for } t = k, k+1, \dots \\ 0 & \text{otherwise} \end{cases}$$

Pascal PMF.

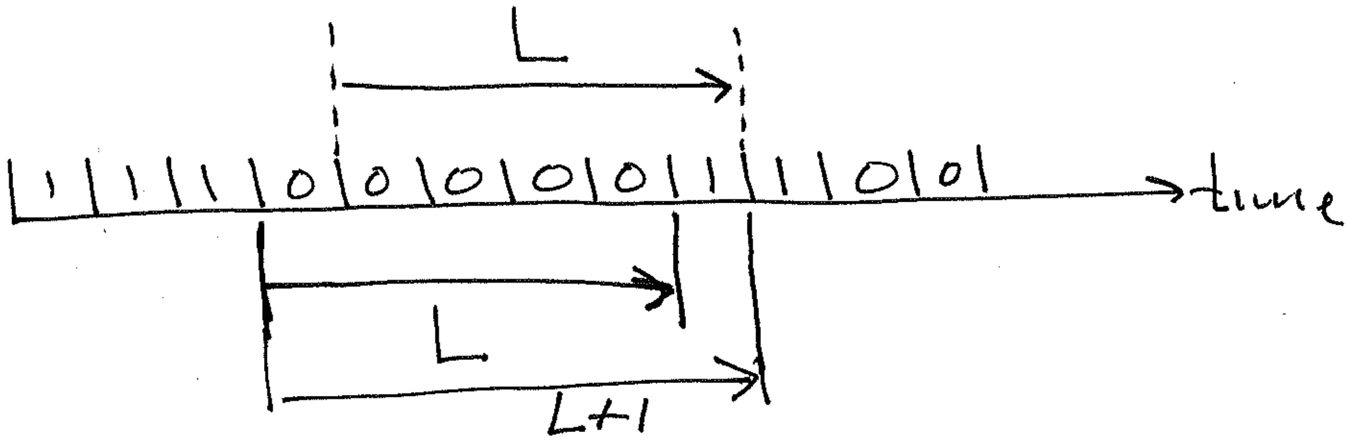
Also

$$E[Y_k] = \frac{k}{p}$$

$$Var(Y_k) = \frac{k(1-p)}{p^2}.$$

Ex.

You buy a lottery ticket every day. let L be length of your 1^{st} losing streak.



What is the distribution of L?

might think: $L+1$ is geometric.

This is wrong: L cannot be 0, so $L+1$ cannot be 1, so $L+1$ cannot be geometric

L is also the time from 2^{nd} loss to next win.

$\therefore$ L is geometric (p).

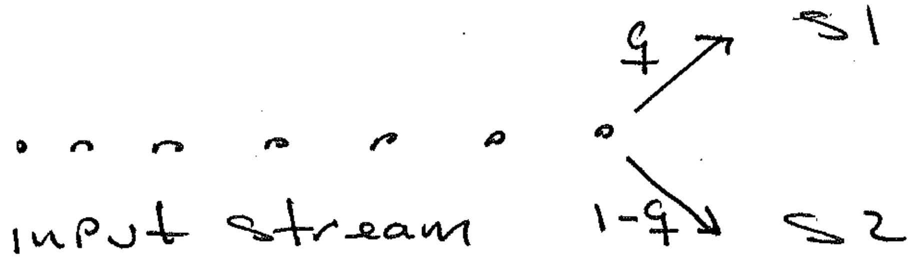
Splitting a Bernoulli Process

Suppose we have a B.P. (Param. p) producing a stream of arrivals of jobs at a server queue. Each job is sent to one of 2 servers

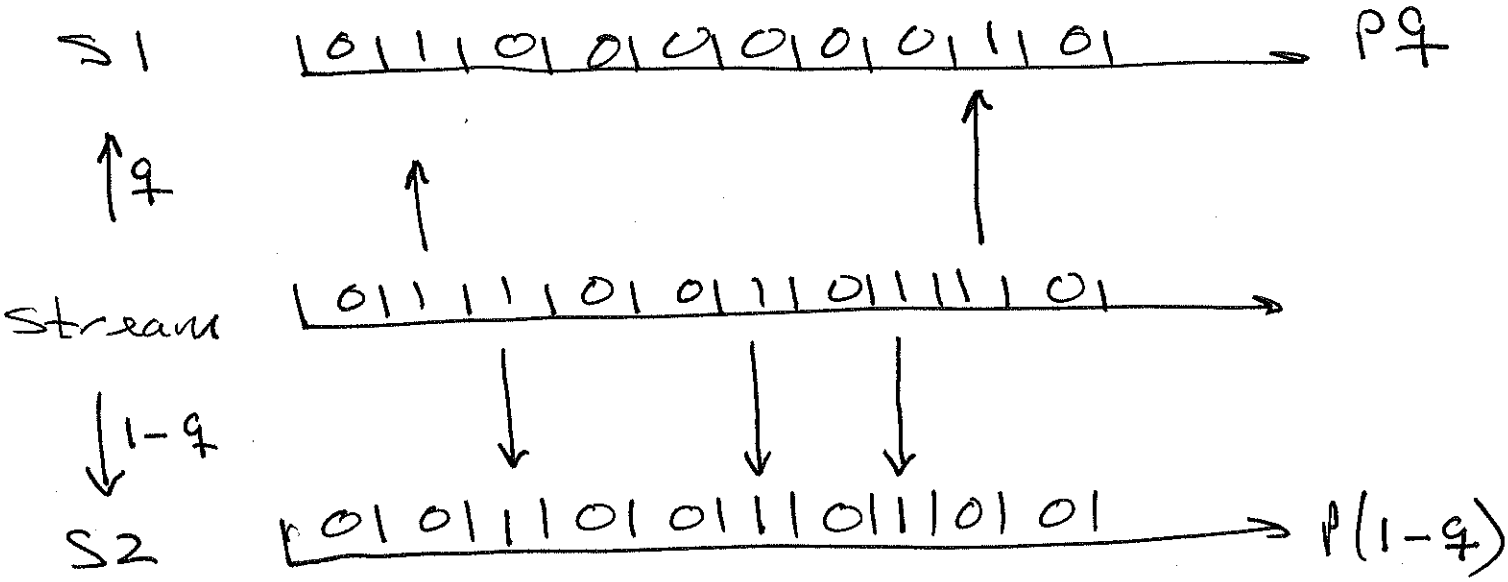
$$P(S1 \text{ gets Job}) = q$$

$$P(S2 \text{ gets Job}) = 1 - q$$

independently of input stream.



What kind of Process is seen by each Server.



events in S1 are all ind. since same time events are ind. in input stream. likewise for S2

Thus

S1 : Bernoulli Process : pq

and

S2 : " " : $p(1-q)$

Merging two Bernoulli Processes

Suppose we're given two independent simultaneous B.P. s

	Param.
stream 1: $X_1, X_2, \ldots$	p
stream 2: $Y_1, Y_2, \ldots$	q

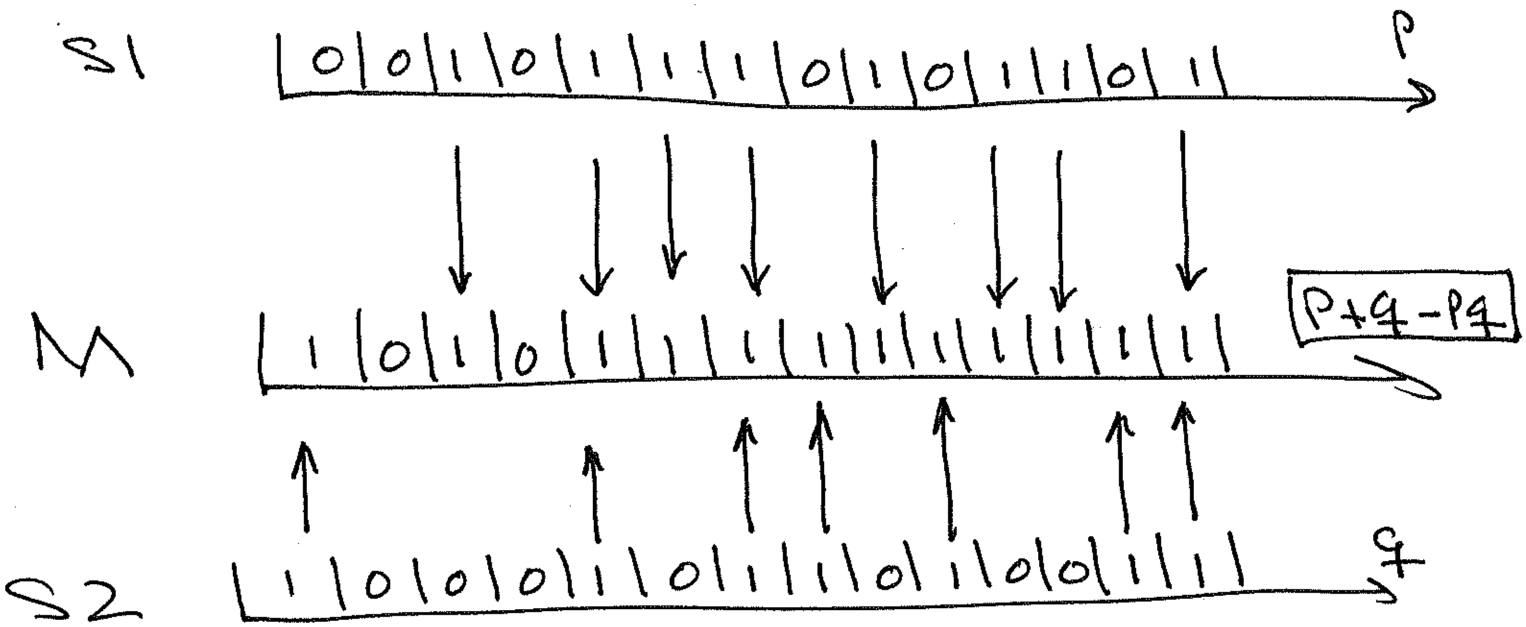
we merge the events of S1 and S2 into a new Process

$$M = (M_1, M_2, \ldots\ldots)$$

according to

$$M_t = \begin{cases} 1 & \text{if either } X_t = 1, \text{ or } Y_t = 1 \\ 0 & \text{if both } X_t = 0, \text{ and } Y_t = 0 \end{cases}$$

what kind of Process is M ?



Observe

$$P(M_t = 1) = 1 - P(M_t = 0)$$

$$= 1 - P(X_t = 0, Y_t = 0)$$

$$= 1 - P(X_t = 0)P(Y_t = 0)$$

$$= 1 - (1-p)(1-q)$$

$$= 1 - (1 - p - q + pq)$$

$$= p + q - pq$$

so M is Bernoulli Process with Param. $p+q-pq$.

<u>Exercise</u>

- first split, then merge. what kind of Process results?
- first merge, then split. what kind of Processes result?
- we merged using logical <u>or</u>. try merging with <u>and</u>, <u>exor</u>, <u>nand</u>, <u>nor</u> and determine the resulting Process in each case.