CSE 107 11-19-22

Supplemental lecture

Variance of a sum

let X, Y be r.v.s. Then

$$\mathrm{Var}(X+Y)$$

$$= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y)$$

more generally

$$V\left(\sum_{i=1}^{n} X_i\right)$$

$$= \sum_{i=1}^{n} \mathrm{Var}(X_i) + \sum_{i \neq j} \mathrm{Cov}(X_i, X_j)$$

Proof in case $n=2$

Let

$$U = X - E[X], \quad \therefore E[U] = 0$$

$$V = Y - E[Y], \quad \therefore E[V] = 0$$

Then

$$\mathrm{Var}(X+Y) = \mathrm{Var}(U+V)$$

$$= E\left[\left((U+V)-(0+0)\right)^2\right]$$

$$= E\left[(U+V)^2\right]$$

$$= E\left[U^2 + V^2 + 2UV\right]$$

$$= E[U^2] + E[V^2] + 2E[UV]$$

$$= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y) \quad \blacksquare$$

4.3 cond. expectation & variance

Recall

$$g(y) = E[X \mid Y = y]$$

is a function of y. If we substitute Y for y, we get a new r.v

$$E[X \mid Y] = g(Y)$$

called <u>conditional expectation</u>. Its expected value

$$E\big[E[X \mid Y]\big] = E[g(Y)]$$

$$= \int_{-\infty}^{\infty} g(y) f_Y(y)\, dy \quad \Big\{ \text{by expected value rule}$$

$$= \int_{-\infty}^{\infty} E[X|Y=y] f_Y(y)\,dy$$

$$= E[X]$$ { by total expectation theorem

$$\boxed{E[E[X|Y]] = E[X]}$$

This is: <u>law of iterated expectations</u>

similarly for Variance. define a function

$$h(y) = Var(X \mid Y=y)$$

$$= E\left[(X - E[X|Y=y])^2 \mid Y=y\right]$$

now substitute Y for y to get a new r.v.

$$Var(X|Y) = h(Y)$$

$$= E[(X - E[X|Y])^2 | Y]$$

we call this r.v. the <u>conditional variance</u> of X given Y. we have

<u>law of total variance</u>

$$\boxed{Var(X) = E[Var(X|Y)] + Var(E[X|Y])}$$

see p. 225-227 for proof & some examples.

skip: 4.4

4.5 sum of a random number of r.v.s

Let $X_1, X_2, X_3, \ldots$ be an infinite sequence of independent identically distributed (iid) r.v.s. This means they all have same PDF (or PMF)

$$f_X(x) \quad \text{or} \quad P_X(x).$$

let $E[X]$ & $Var(X)$ be their common mean & variance. Let N be a r.v. with range$(N)=\{0,1,2,\ldots\}$. Assume $\{N, X_1, X_2, \ldots\}$ is independent.

define

$$Y = X_1 + X_2 + \cdots + X_N$$

Then, fix $n \geq 0$:

$E[Y \mid N = n]$

$= E[X_1 + \cdots + X_N \mid N = n]$

$= E[X_1 + \cdots + X_n \mid N = n]$

$= E[X_1 + \cdots + X_n]$ {by independence

$= E[X_1] + \cdots + E[X_n]$

$= n E[X]$.

Thus: $E[Y \mid N] = N E[X]$

by law of iterated expectation:

$$E[Y] = E[E[Y|N]]$$

$$= E[N\,E[X]]$$

$$= E[N] \cdot E[X]$$

Also

$$Var(Y|N=n)$$

$$= Var(X_1 + \dots + X_N | N=n)$$

$$= Var(X_1 + \dots + X_n | N=n)$$

$$= Var(X_1 + \dots + X_n) \quad \{\text{by independence}$$

$$= Var(X_1) + \dots + Var(X_n) \quad \{\text{by independence}$$

$$= n \cdot Var(X)$$

now replace n by N:

$$Var(Y \mid N) = N \cdot Var(X)$$

By law of total variance

$$Var(Y) = E(Var(Y|N)) + Var(E[Y|N])$$

$$= E[N \cdot Var(X)] + Var(N \cdot E[X])$$

$$= E[N] \cdot Var(X) + E[X]^2 \cdot Var(N)$$

<u>EX</u>

Suppose we have 2 coins

coin 1: $P(\text{head}) = p$

coin 2: $P(\text{head}) = q$

Flip coin 1 until first head. Let

$N = \#$ flips

so N is geometric with parameter p. Now flip coin 2 N times, and count heads. Let Y be the # of heads. Note

$$Y = X_1 + X_2 + \cdots + X_N$$

where each X_i is Bernoulli with parameter q.

$$X_i = \begin{cases} 1 & \text{if coin 2 is head} \\ 0 & \text{" " " " tail.} \end{cases}$$

All coin flips are independent.

let $E[X]$ & $Var(X)$ be mean & Variance of Bernoulli with Parameter p.

we have

$$E[N] = \frac{1}{p} \quad , \quad Var(N) = \frac{1-p}{p^2}$$

also

$$E[X] = q \quad , \quad Var(X) = q(1-q)$$

Therefore

$$E[Y] = E[N] \cdot E[X] = \frac{q}{p}$$

and

$$Var(Y) = E[N] \cdot Var(X) + E[X]^2 \cdot Var(N)$$

$$= \frac{q(1-q)}{p} + \frac{q^2(1-p)}{p^2}$$