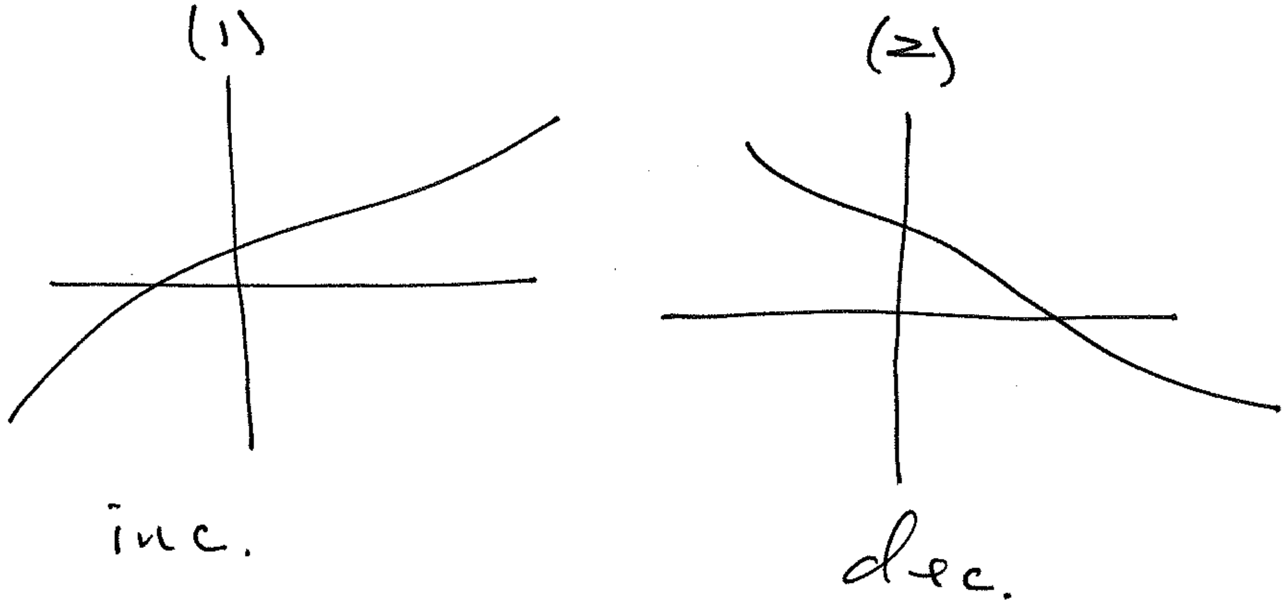
cse 107 11-15-22

monotonic fcns



a strictly monotone fcn is invertible. let $g:\mathbb{R}\to\mathbb{R}$ be strictly monotone, and $h:\mathbb{R}\to\mathbb{R}$ its inverse $h=g^{-1}$

Then

$(1) \iff g'>0 \iff h'>0$

and $(2) \iff g'<0 \iff h'<0$

let X be a cont. r.v. and

$$Y = g(X)$$

Then

$$f_Y(y) = f_X(h(y)) \cdot |h'(y)|$$

Proof

Suppose g is inc., and $\therefore$ so is h.

Then

$$
\begin{aligned}
F_Y(y) &= P(Y \le y) \\
&= P(g(X) \le y) \\
&= P(X \le h(y)) \\
&= F_X(h(y))
\end{aligned}
$$

now diff w.r.t. y :

$$f_Y(y) = f_X(h(y)) \cdot h'(y)$$

$$= f_X(h(y)) \cdot |h'(y)| \quad \left\{ \begin{array}{l} \text{since} \\ h' > 0. \end{array} \right.$$

Exercise: do case: g is decreasing.

$\blacksquare$

Ex.

let $Y = X^2$ where X is a cont. r.v. with range$(X) \subseteq [0, \infty)$.

Then also range$(Y) \subseteq [0, \infty)$

note: $g(x) = x^2$ is mono. strictly inc. on $[0, \infty)$, and so is its inverse $h(y) = \sqrt{y}$. Thus

$$f_Y(y) = f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} \qquad (y > 0)$$

<u>Two r.v.s</u> $\quad Z = g(X, Y)$

<u>Ex.</u> let X, Y be ind., both unif. on $[0, 1]$.

$$\therefore \quad f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

and $F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$

define $Z = \max(X, Y)$. Then

for $0 \le z \le 1$:

$$F_Z(z) = P(Z \le z)$$

$$= P(\max(X,Y) \le z)$$

$$= P(X \le z, Y \le z)$$

$$= P(X \le z) \cdot P(Y \le z) \quad \Big\{ \text{by ind.}$$

$$= F_X(z) \cdot F_Y(z)$$

$$= z \cdot z = z^2$$

$\therefore$

$$f_Z(z) = \begin{cases} 2z & 0 \le z \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Sums & convolutions

let X, Y be ind. discrete r.v.s

let

$$Z = X + Y$$

for any $z \in \text{range}(Z)$:

$$p_Z(z) = P(X + Y = z)$$

$$= \sum_{\{(x,y) \mid x+y=z\}} P(X=x, Y=y)$$

$$= \sum_{\{(x,z-x) \mid x \in \text{range}(X)\}} P(X=x) \cdot P(Y=z-x) \quad \{\text{by ind.}$$

$$= \sum_{x} P(X=x) \cdot P(Y=z-x)$$

$$= \sum_{x} P_X(x) P_Y(z-x)$$

$$\therefore P_Z(z) = \underbrace{\sum_{x} P_X(x) \cdot P_Y(z-x)}_{\text{Convolution Product of } P_X \text{ and } P_Y.}$$