cse 107 11-10-22

Continuous Baye's rule:

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) \cdot f_X(x)}{\int_{-\infty}^{\infty} f_{Y|X}(y|t) \cdot f_X(t)\, dt}$$

<u>Ex.</u>

Let Y be a normal r.v. with s.d. 1 and mean X, another r.v. X is uniform on $[1,3]$

a.) find $f_Y(y)$

<u>soln</u>.

we are given

$$f_X(x) = \begin{cases} \frac{1}{2} & 1 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{\sqrt{2\pi}} \cdot e^{-(y-x)^2/2} & \begin{array}{l} 1 \leq x \leq 3 \\ y \in \mathbb{R} \end{array} \\ 0 & \text{otherwise} \end{cases}$$

so

$$f_Y(y) = \int_{-\infty}^{\infty} f_{Y|X}(y|x) \cdot f_X(x)\, dx$$

$$= \frac{1}{2} \int_{1}^{3} \frac{1}{\sqrt{2\pi}} \cdot e^{-(x-y)^2/2} \, dx \qquad \begin{array}{l} u = x-y \\ du = dx \end{array}$$

$$= \frac{1}{2} \int_{1-y}^{3-y} \frac{1}{\sqrt{2\pi}} \cdot e^{-u^2/2} \, du$$

$$= \frac{1}{2} \left(\Phi(3-y) - \Phi(1-y) \right)$$

b.) find $f_{X|Y}(x|y)$

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) \cdot f_X(x)}{f_Y(y)}$$

$$= \begin{cases} \dfrac{\frac{1}{\sqrt{2\pi}} e^{-(y-x)^2/2}}{\Phi(3-y) - \Phi(1-y)} & 1 \leq x \leq 3 \\ & y \in \mathbb{R} \\ 0 & \text{otherwise} \end{cases}$$

c.) suppose we sample Y and get $y=3$. what is the probability that $X \leq 2$?

$$P(X \leq 2 \mid Y=3) = \int_1^2 f_{X|Y}(x|3)\,dx$$

$$= \frac{\int_1^2 \frac{1}{\sqrt{2\pi}} e^{-(x-3)^2/2}\,dx}{\Phi(0) - \Phi(-2)}$$

$$= \frac{\int_{-2}^{-1} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \, du}{\Phi(0) - \Phi(-2)}$$

$$= \frac{\Phi(-1) - \Phi(-2)}{\Phi(0) - \Phi(-2)}$$

$$= \boxed{.2848}$$

d.) find $E[Y]$

$$E[Y] = \int_{-\infty}^{\infty} y \cdot f_Y(y) \, dy$$

$$= \int_{-\infty}^{\infty} y \cdot \frac{1}{2} \cdot \int_{1}^{3} \frac{1}{\sqrt{2\pi}} e^{-(x-y)^2/2} \, dx \, dy$$

$$= \frac{1}{2} \int_1^3 \int_{-\infty}^{\infty} y \cdot \frac{1}{\sqrt{2\pi}} e^{-(y-x)^2/2} \, dy \, dx$$

$$= \frac{1}{2} \int_1^3 x \, dx = \frac{1}{2} \frac{x^2}{2} \Big|_1^3 = \frac{1}{4}(9-1) = \boxed{2}.$$

4.1 Derived distributions

If $Y = g(X)$, how can we find

$$f_Y(y)$$

from

$$f_X(x).$$

We call $f_Y(y)$ a <u>derived dist.</u>

2-step process

1.) find CDF: $F_Y(y)$

$$F_Y(y) = P(g(X) \le y)$$

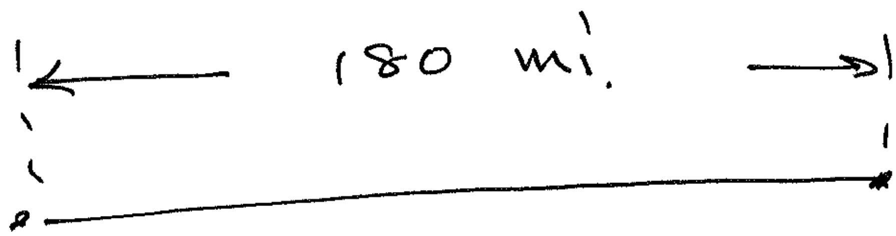
$$= \int_{\{x \mid g(x) \le y\}} f_X(x)\, dx$$

2.) differentiate!

$$f_Y(y) = \frac{d}{dy} F_Y(y).$$

Ex.

A car travels a dist. of 180 miles at const. rate R. Also R is itself a random variable, uniform cont. on $30 \le r \le 60$ (mph)



Let T be the time of travel. Find $f_T(t)$.

Soln.

have: $180 = R \cdot T$

$$\therefore \quad T = \frac{180}{R}$$

(1) $F_T(t) = P(T \le t)$

$$= P\left(\frac{180}{R} \le t\right)$$

$$= P\left(R \ge \frac{180}{t}\right)$$

$$= 1 - P\left(R < \frac{180}{t}\right)$$

$$= 1 - F_R\left(\frac{180}{t}\right)$$

We are given

$$f_R = \begin{cases} \frac{1}{30} & 30 \le r \le 60 \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore F_R(r) = \int_{-\infty}^{r} f_R(s)\, ds$$

$$= \int_{30}^{r} \frac{1}{30}\, ds$$

$$= \frac{1}{30} \cdot s \Big|_{30}^{r} = \frac{r-30}{30}$$

$$= \begin{cases} 0 & r < 30 \\ \frac{r-30}{30} & 30 \leq r \leq 60 \\ 1 & r > 60 \end{cases}$$

hence

$$F_T(t) = 1 - F_R\left(\frac{180}{t}\right)$$

$$= \begin{cases} 0 & t < 3 \\ 2 - \frac{6}{t} & 3 \leq t \leq 6 \\ 1 & t > 6 \end{cases}$$

(2)

$$f_T(t) = \begin{cases} \dfrac{6}{t^2} & 3 \le t \le 6 \\ 0 & \text{otherwise} \end{cases}$$

If $Y = aX + b$, then

$$\boxed{f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)} \qquad (a \neq 0)$$

<u>Proof:</u>

(1)

$$F_Y(y) = P(Y \le y)$$

$$= P(aX + b \le y)$$

$$= \begin{cases} P\left(X \le \frac{y-b}{a}\right) & a > 0 \\ P\left(X \ge \frac{y-b}{a}\right) & a < 0 \end{cases}$$

$$= \begin{cases} P\left(X \le \frac{y-b}{a}\right) & a > 0 \\ 1 - P\left(X < \frac{y-b}{a}\right) & a < 0 \end{cases}$$

$$= \begin{cases} F_X\left(\frac{y-b}{a}\right) & a > 0 \\ 1 - F_X\left(\frac{y-b}{a}\right) & a < 0 \end{cases}$$

2 diff w.r.t. x

$$f_Y(y) = \begin{cases} f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a} & a > 0 \\ -f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a} & a < 0 \end{cases}$$

$$= \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) \quad \blacksquare$$

Ex.

Let X be normal with mean μ and var. σ^2. Then

$$Y = aX + b$$

is also normal with mean $a\mu + b$ and var. $a^2\sigma^2$

Proof: we have

$$f_X(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} \cdot e^{-(x-\mu)^2/2\sigma^2}$$

Thus

$$f_Y(y) = \frac{1}{|a|} \cdot f_X\left(\frac{y-b}{a}\right)$$

$$= \frac{1}{|a|} \cdot \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\left(\frac{y-b}{a} - \mu\right)^2 / 2\sigma^2}$$

$$= \frac{1}{\sqrt{2\pi} \cdot |a|\sigma} \cdot e^{-\left(\frac{y-b-a\mu}{a}\right)^2 / 2\sigma^2}$$

$$= \frac{1}{\sqrt{2\pi} \cdot |a|\sigma}\, e^{-(y-(a\mu+b))^2 / 2(a\sigma)^2}$$

which is normal. $\blacksquare$

Defn

A function $g : \mathbb{R} \to \mathbb{R}$ is called <u>strictly monotonic</u> iff for all $x_1, x_2 \in \mathbb{R}$, either

(1) $x_1 < x_2 \Rightarrow g(x_1) < g(x_2)$ (<u>increasing</u>)

or

(2) $x_1 < x_2 \Rightarrow g(x_1) > g(x_2)$ (<u>decreasing</u>).

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