- midterm 1 graded
- hw5 due Thur.

Exercise:

compute CDFs of all 9 random variables in r.v. handout.

3.3 Normal random variables

We call X normal or Gaussian if it has PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma}\cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Parameters: μ, $\sigma > 0$

It can be shown

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1 .$$ { see #14 in text

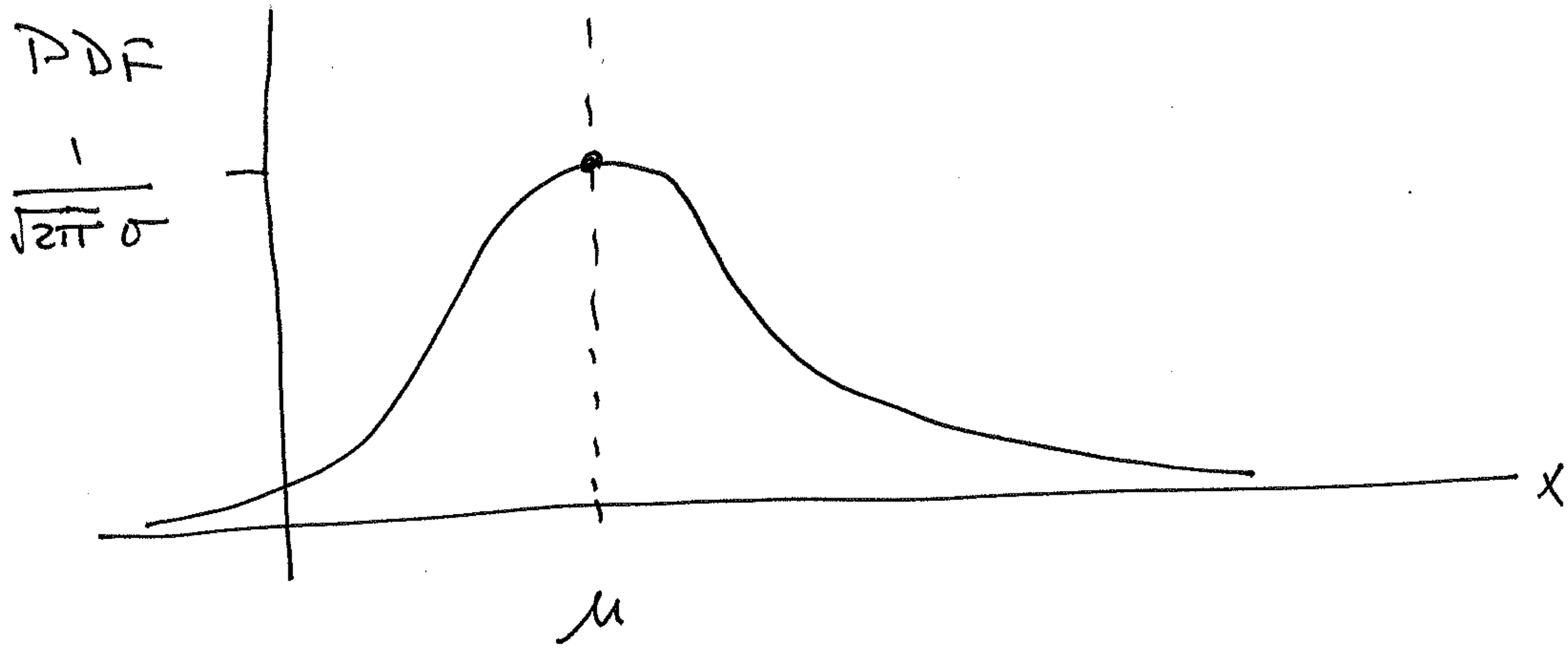
so this is a valid PDF.

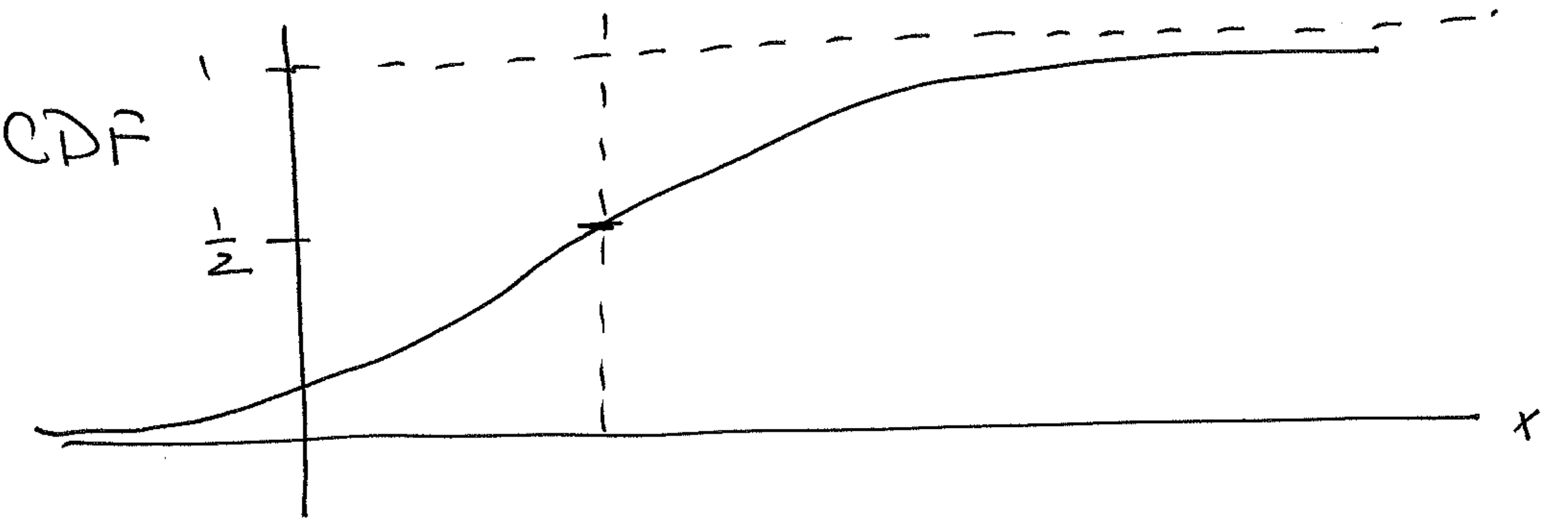
Also

$$E[X] = \mu$$

$$Var(X) = \sigma^2$$

$$\therefore \sqrt{Var(X)} = \sigma$$





<u>note</u>: $f_X(x)$ is invariant under the transformation

$$(x-\mu) \longleftrightarrow -(x-\mu)$$

so $E[X] = \mu$.

see handout for other identities.

<u>Theorem</u>

If X is a normal r.v. with μ, σ as parameters, then

$$Y = aX + b$$

is also <u>normal</u> with

mean: $E[Y] = a\mu + b$

variance: $Var(Y) = a^2 \cdot \sigma^2$

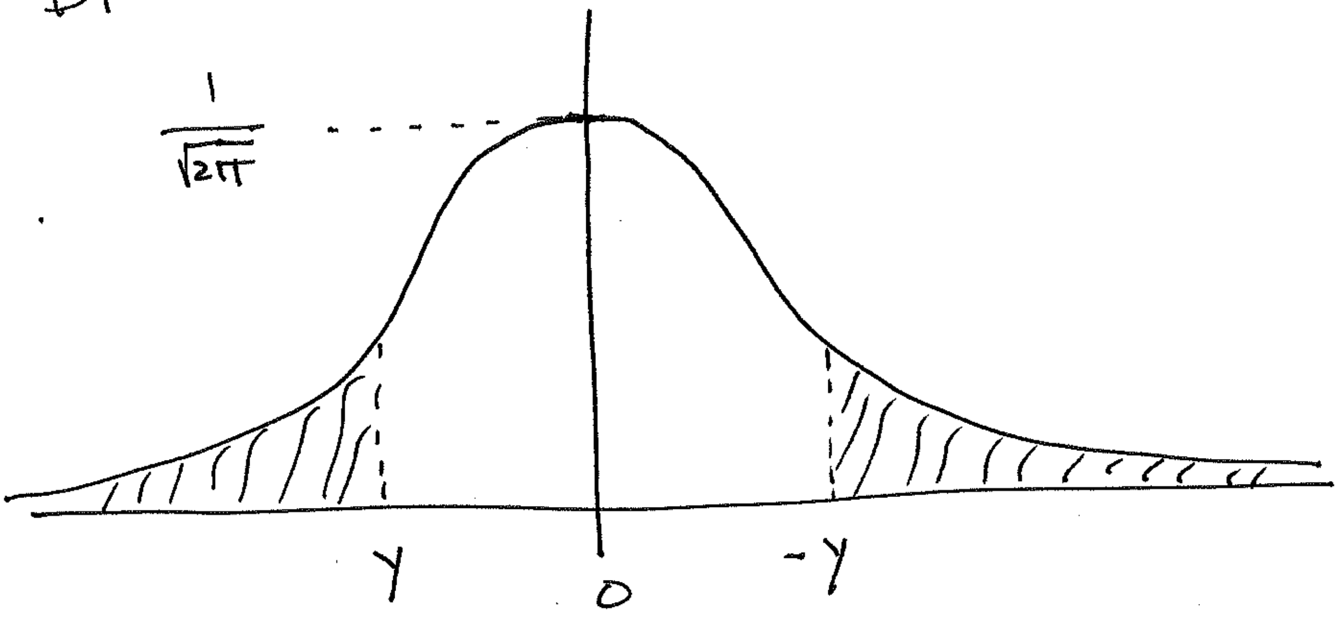
If Y ^a normal r.v. has mean 0 and variance 1, we call it <u>standard normal</u>.

PDF : $f_Y(y) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}}$

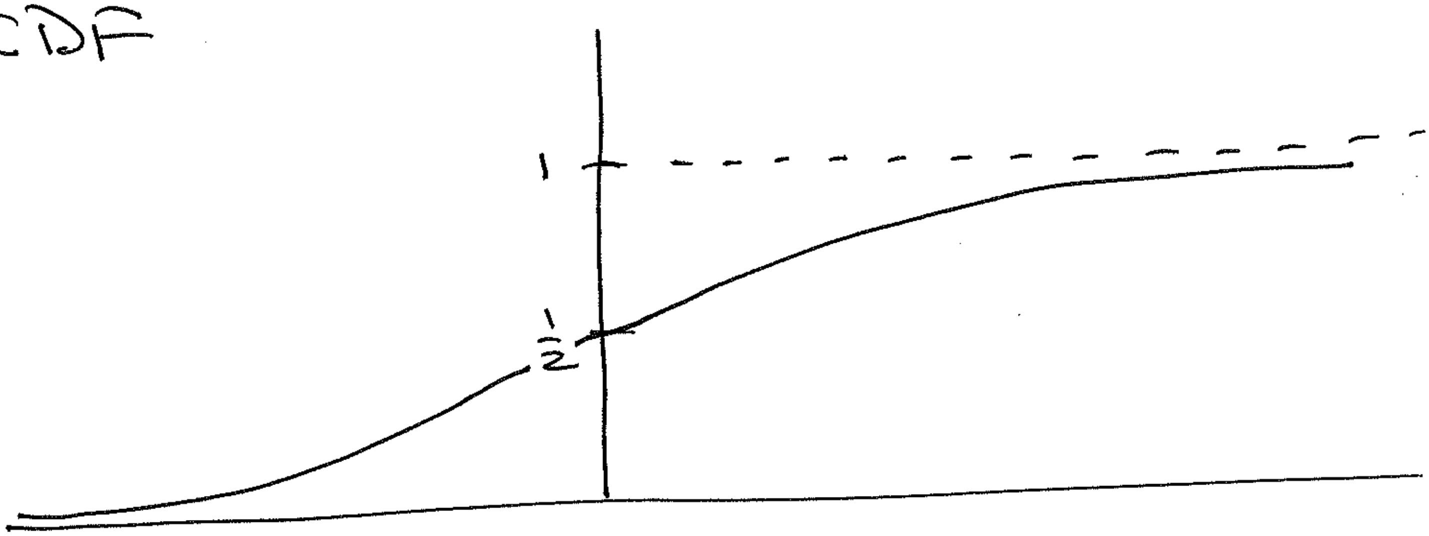
The CDF of Y is denoted

$$\Phi(y) = P(Y \le y)$$
$$= P(Y < y)$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{t^2}{2}} \, dt$$

PDF



CDF



handout: std. normal table $\Phi(y)$ for $0 \le y \le 3.9$. note identity

$$\Phi(-y) = 1 - \Phi(y)$$

allows us to compute $\Phi(y)$ for $y < 0$.

Ex.

- $\Phi(1.84) = .9671$
- $\Phi(-1.84) = 1 - (.9671) = .0329$

We compute values for $F_X(x)$ where X is a normal r.v. with mean μ, and s.d. σ as follows.

let

$$Y = \frac{X-\mu}{\sigma} = \frac{1}{\sigma}X - \frac{\mu}{\sigma}$$

Thus

$$E[Y] = \frac{1}{\sigma}\mu - \frac{\mu}{\sigma} = \frac{\mu-\mu}{\sigma} = 0$$

and

$$Var(Y) = \frac{1}{\sigma^2}Var(X)$$

$$= \frac{1}{\sigma^2}\cdot\sigma^2 = 1.$$

$\therefore$ Y is std normal.

The

$$F_X(x) = P(X \le x)$$

$$= P(X - \mu \le x - \mu)$$

$$= P\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right)$$

$$= P\left(Y \le \frac{x-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{x-\mu}{\sigma}\right)$$

<u>Ex.</u>

The mean high temp. in Santa Cruz on Nov. 1 is 66° F with s.d. 4.4°. What's the probability that todays high temp. is $\le 57^\circ$.

let X be todays high temp and assume X is normal.

Soln

$$P(X \le 57) = P\left(\frac{X-66}{4.4} \le \frac{57-66}{4.4}\right)$$

$$= P(Y \le -2.04)$$

$$= \Phi(-2.04)$$

$$= 1 - \Phi(2.04)$$

$$= 1 - .9793$$

$$= \boxed{.0207}$$

Ex.

what is Prob. that a normal r.v. is within 1 standard deviation of its mean?

Soln

$$P(\mu-\sigma \le X \le \mu+\sigma)$$

$$= P\left(\frac{(\mu-\sigma)-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{(\mu+\sigma)-\mu}{\sigma}\right)$$

$$= P(-1 \le Y \le 1)$$

$$= P(Y \le 1) - P(Y < -1)$$

$$= \Phi(1) - \Phi(-1)$$

$$= \Phi(1) - (1 - \Phi(1))$$

$$= 2\Phi(1) - 1$$

$$= 2(.8413) - 1 = \boxed{.6826}$$

for 2σ we get:

$$2\Phi(2) - 1 = 2(.9772) - 1 = \boxed{.9544}$$

for 3σ we get

$$2\Phi(3) - 1 = 2(.9987) - 1 = \boxed{.9974}$$

3.4 Joint PDFs of multiple r.v.s

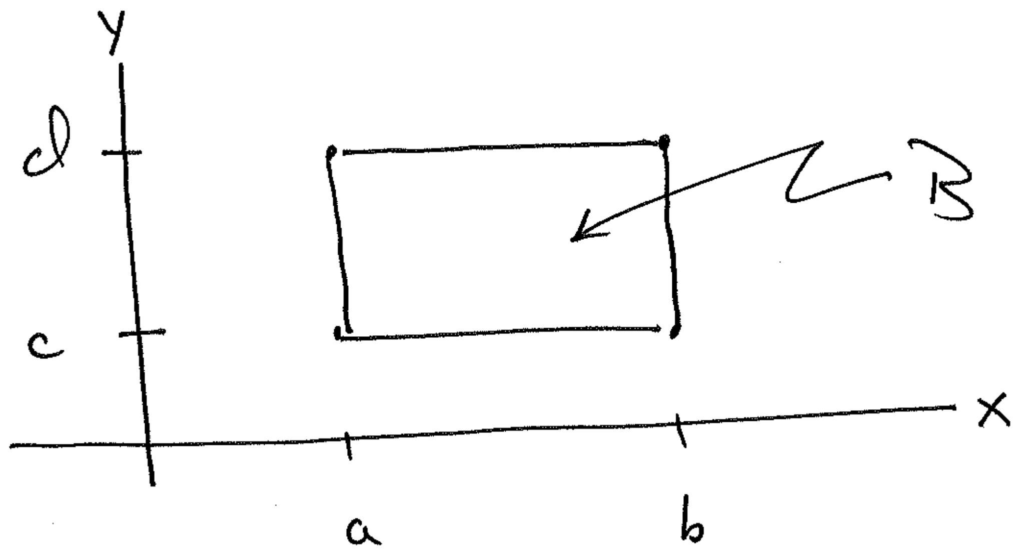
<u>Defn</u>

We say that r.v.s X, Y are <u>jointly continuous</u> iff there exists a function $f_{X,Y}(x,y)$ s.t.

$$P\big((X,Y)\in B\big) = \iint_B f_{X,Y}(x,y)\,dx\,dy$$

for 'any' subset $B \subseteq \mathbb{R}^2$. The function $f_{X,Y}(x,y)$ is called the <u>joint PDF</u> of X, Y.

If $B = [a,b] \times [c,d]$



then

$$P(a \le X \le b,\ c \le Y \le d)$$

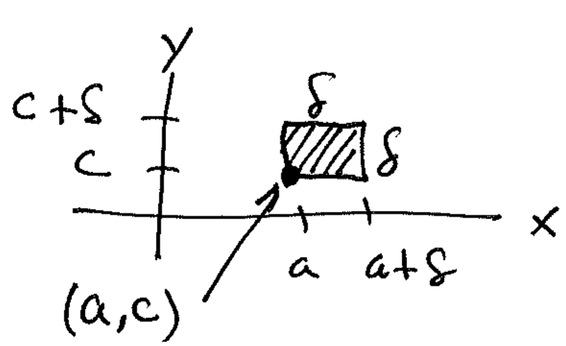
$$= \int_a^b \int_c^d f_{X,Y}(x,y)\, dy\, dx$$

If $B = \mathbb{R}^2$, we must have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = P(X \in \mathbb{R}, Y \in \mathbb{R})$$
$$= P(\Omega)$$
$$= 1$$

If $\delta > 0$ is small, and $f_{X,Y}(x,y)$ is continuous near (a,c), then

$$P(a \le X \le a+\delta,\ c \le Y \le c+\delta)$$
$$= \int_a^{a+\delta}\int_c^{c+\delta} f_{X,Y}(x,y)\,dy\,dx$$



$$\approx f_{X,Y}(a,c) \cdot \delta^2$$

$\therefore$ $f_{X,Y}(a,c)$ has units:

$$\frac{\text{Probability}}{\text{unit area}}$$

i.e. mass density in 2-dim.

Recall if $A \subseteq \mathbb{R}$, we have

$$P(X \in A) = \int_A f_X(x)\,dx$$

But also

$$P(X \in A) = P(X \in A, Y \in \mathbb{R})$$

$$= \iint_{A \times \mathbb{R}} f_{X,Y}(x,y)\, dy\, dx$$

$$= \int_A \left(\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \right) dx$$

Thus

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

similarly

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

Ex. Joint uniform continuous on $R = [a,b] \times [c,d]$

$$f_{X,Y}(x,y) = \begin{cases} \alpha & (x,y) \in R \\ 0 & \text{otherwise} \end{cases}$$

we must have

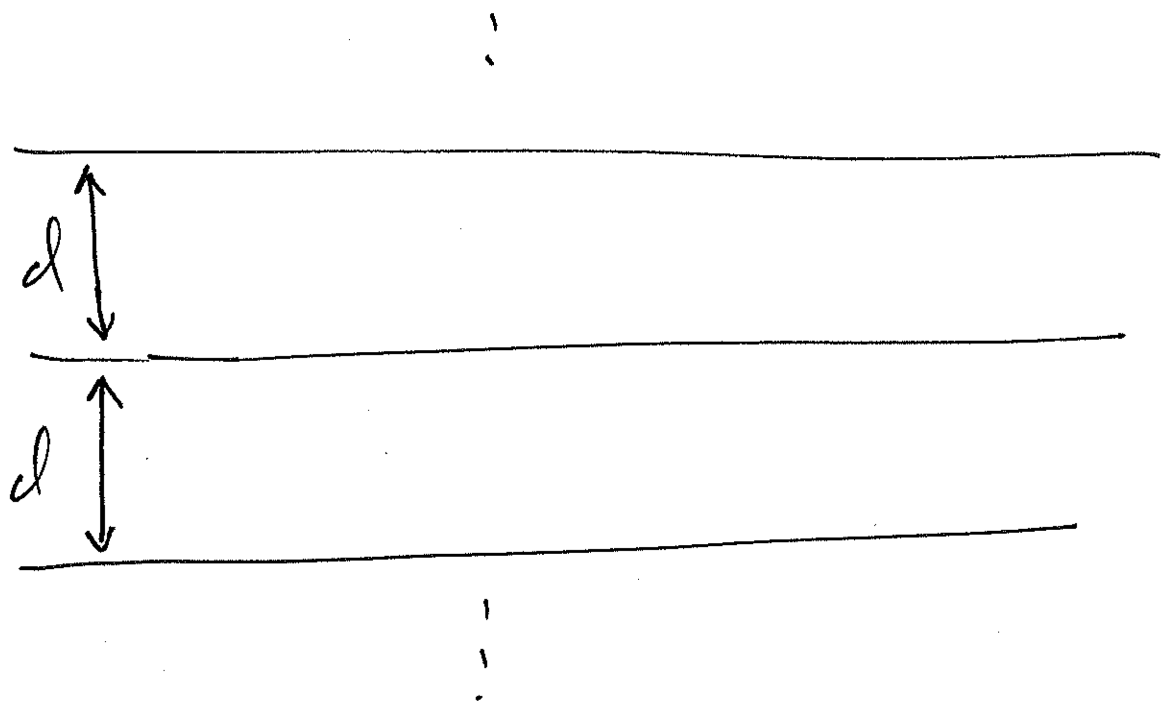
$$1 = \iint_{\mathbb{R}^2} f_{X,Y}(x,y)\,dx\,dy = \iint_R \alpha\,dx\,dy$$

$$= \alpha \cdot \iint_R dx\,dy = \alpha \cdot \text{area}(R)$$

$$\therefore \quad \alpha = \frac{1}{\text{area}(R)} = \frac{1}{(b-a)\cdot(d-c)}$$

Ex. Buffon's needle

A surface is ruled with parallel lines dist. d apart



a needle of length l is tossed on the surface (assume $l < d$)

what is the prob. that the needle intersects one of the lines?